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WITH THEIR

SOLUTIONS.

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AND THREE

APPENDICES

CONTAINING MANY FURTHER PAPERS AND SOLUTIONS.

EDITED BY

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Mathematical Papers, &c.

Questions Solved.

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES:

WITH ADDITIONAL PAPERS AND SOLUTIONS.

8115. (By Professor SYLVESTER, F.R.S.)—If $U = 0$, $V = 0$ be the equations to two general surfaces of the degrees m, n respectively in the variables, and if $I = 0$ express the relation between the coefficients of U and V when these two surfaces touch; prove that I will be of the degrees

$$n\left[(n-1)^2 + 2(m-1)(n-1) + 3(m-1)^2\right],$$
$$m\left[(m-1)^2 + 2(n-1)(m-1) + 3(n-1)^2\right],$$

in regard of the coefficients of U and V respectively.

Solution by SAMUEL ROBERTS, M.A.

(See SALMON's *Geometry of Three Dimensions*, 3rd ed., p. 514; also *Quarterly Journal of Mathematics*, Vol. XII., p. 229, *et seq.*)

Let x_1, x_2, x_3, x_4 be the four current homogeneous coordinates. To obtain the weight of the correlation-function I in the coefficients, we may take the gross system

$$\frac{dU}{dx_1}\cdot\frac{dV}{dx_2}-\frac{dU}{dx_2}\cdot\frac{dV}{dx_1}=0,\ U=0,\quad \frac{dU}{dx_1}\cdot\frac{dV}{dx_3}-\frac{dU}{dx_3}\cdot\frac{dV}{dx_1}=0,\ V=0\ldots\text{(A)},$$

and deduct from the weight of the resultant the weights of the resultants of the systems

$$\frac{dU}{dx_1}=\frac{dV}{dx_1}=U=V=0\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)},$$

$$\frac{dU}{dx_1}\frac{dV}{dx_2}-\frac{dU}{dx_2}\frac{dV}{dx_1}=\frac{dU}{dx_1}\frac{dV}{dx_3}-\frac{dU}{dx_3}\frac{dV}{dx_1}=U=x_4=0\ldots\ldots\ldots\text{(C)}.$$

But in the case of (C) we must exclude the extraneous systems

$$\frac{dU}{dx_1}=\frac{dV}{dx_1}=U=x_4=0,\quad \frac{dU}{dx_1}=\frac{dU}{dx_2}=\frac{dU}{dx_3}=x_4=0.$$

Now, if the coefficients of U contain a variable in the degree μ, and those of V contain the variable in the degree ν, the order of the resultant of (A) in this variable is

$$(m+n-2)^2(n\mu+m\nu)+2mn(m+n-2)(\mu+\nu).$$

For (B) we have

$$mn\left[(m-1)\nu+(n-1)\mu\right]+n(m-1)(n-1)\mu+m(m-1)(n-1)\nu;$$

and for (C) we have $(2mn+n^2-3n)\mu+(2mn+m^2-3m)\nu$.

The required order is, Order for (A)—Order for (B)—Order for (C),

or $(3m^2+n^2+2mn-4n-8m+6)n\mu+(3n^2+m^2+2mn-4m-8n+6)m\nu$,

and we may make $\mu=1$, $\nu=0$, and $\mu=0$, $\nu=1$, in succession.

[For two *curves* U, V of the degrees m, n, the degree of I (when $I=0$, expresses the condition of their contact) has been found by Professor CAYLEY to be $n(n+2m-3)$, $m(m+2n-3)$, in respect of the two sets of coefficients, and he has proposed the name of Tact-Invariant for the function I. Discriminants, Tact-Invariants, and Resultants are only special cases of a well-marked family of combinantive invariants, proposed to be termed Osculants. For the definition and degree of any osculant whatever, see a *Note* by the Proposer in the *Comptes Rendus de l'Institut*.

The PROPOSER remarks that the expressions are, respectively,

for curves, $\quad m[(m-1)+2(n-1)]$;

for surfaces, $\quad m[(m-1)^2+2(n-1)(m-1)+3(n-1)^2]$;

and, for the next case,

$m[(m-1)^3+2(n-1)(m-1)^2+3(n-1)^2(m-1)+4(n-1)^3]$, and so on.

The degree of a general Tact-Invariant to the two quantics of the degrees m, n respectively in i variables is, calling $m-1=\mu$ and $n-1=\nu$, in the

U coefficients, $\quad n\dfrac{\nu^i-i\nu\mu^{i-1}+(i-1)\mu^i}{(\nu-\mu)^2}$,

and in the V coefficients, $m\dfrac{\mu^i-i\mu\nu^{i-1}+(i-1)\nu^i}{(\mu-\nu)^2}$,

the series giving the degrees being obviously summable under this form.]

8081. (By Professor WOLSTENHOLME, M.A., Sc.D.)—In a tetrahedron ABCD, the lengths of the edges DA, DB, DC are denoted by a, b, c, and the lengths of the edges BC, CA, AB by a', b', c'; also the dihedral angles respectively *opposite* to these edges are denoted by $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$: prove that (1), if $a'+b'+c'>a'+b+c>a+b'+c>a+b+c'$, then

$$\alpha'+\beta'+\gamma'>\alpha'+\beta+\gamma>\alpha+\beta'+\gamma>\alpha+\beta+\gamma';$$

and (2), if $a+a'>b+b'>c+c'$, then $\alpha+\alpha'>\beta+\beta'>\gamma+\gamma'$;

also find (3) how many different orders of magnitude of the quantities a, b, c, a', b', c' will be possible, when the above five inequalities hold good; and (4) under what conditions the order of magnitude of a, b, c, a', b', c' will be the same as that of the respectively opposite dihedral angles.

Solution by D. BIDDLE.

1. By reduction, we have (i.) $b_1+c_1>b+c$, (ii.) $a_1+c_1>a+c$, (iii.) $a_1+b_1>a+b$, (iv.) $a_1+b>a+b_1$, (v.) $a_1+c>a+c_1$, (vi.) $b_1+c>b+c_1$;

from (iii.), (iv.), we have $a_1 > a$; and from (i.), (vi.), $b_1 > b$; and from these conjoined, $a_1 - a > b_1 - b > c_1 \sim c$, in which case, also, $a_1 - a > \beta_1 - \beta > \gamma_1 \sim \gamma$. Now, it has already been proved, in the solution to Quest. 7425 (Vol. xl., p. 122), that, if ABCD be a tetrahedron in which $AB + AC = DB + DC$, then $\widehat{AB} + \widehat{AC} = \widehat{DB} + \widehat{DC}$, where $\widehat{AB}$ is the dihedral angle between the planes meeting in AB. Let us then take the inequality marked (i.) above, in which $CA + AB > CD + DB$. Extend CD to D_1, so that

$$CD_1 + D_1B = CA + AB,$$

and join AD_1. Then it is clear that the dihedral angle $\widehat{AB}$ is enlarged; let it $= \gamma + x$. It is also clear that $\widehat{DB} > \widehat{D_1B}$; for, if a perpendicular be drawn from A to meet the plane of BCD in P, and PQ, PR be drawn perpendicular to BD, BD_1, then AQP indicates, either directly or as the supplement, according to the position of P, a dihedral angle which is larger than that indicated by ARP. Let $\widehat{D_1B} = \beta_1 - x_1$; then we have

$$(\beta_1 - x_1) + \gamma_1 = \beta + (\gamma + x), \quad \text{whence} \quad \beta_1 + \gamma_1 > \beta + \gamma, \text{ by } x + x_1.$$

The other five inequalities can be treated in the same way.

2. Having found that, when $a_1 - a > b_1 - b > c_1 \sim c$, $a_1 - a > \beta_1 - \beta > \gamma_1 \sim \gamma$, we can in the same way prove that, when $a_1 + a > b_1 + b > c_1 + c$, then $a_1 + a > \beta_1 + \beta > \gamma_1 + \gamma$. In the present case, $a_1 + b_1$ may equal $a + b$, or, again, $a_1 + b$ may equal $a + b_1$, and the corresponding pairs of dihedral angles (those opposite) be also equal, according to Quest. 7425 in Vol. xl., and we will begin with the limiting instance, in which $a_1 = b$, and $a = b_1$. Here it is plain that, if the two faces be imagined capable of rotation about their common edge until in opposition, they will exactly coincide, and that $a_1 = \beta$, and $a = \beta_1$; *i.e.*, $a_1 + a = \beta_1 + \beta$. If now, maintaining three of the four edges in their original position, we alter the direction of a_1 or a, by lessening the angle it forms with the common edge of the two faces, then $a_1 + a$ is clearly greater than $b_1 + b$, and at the same time the dihedral angle at the changed edge is greater than before, whilst that formed by the changed plane with the other face is less than before. Hence, in this instance, $a_1 + a > \beta_1 + \beta$. The lengthening of a (or a_1) and shortening of b (or b_1) can be effected also by moving the point of their intersection, in the direction a to b (or a_1 to b_1), along the ellipse which is the locus of equality in their sum, and the foci of which are the extremities of the common edge of the two faces; and in this case it is perfectly clear that $a_1 > \beta$ and $a > \beta_1$. And in all other cases, by producing b, if $a_1 + b_1 > a + b$, or, by producing b_1, if $a + b > a_1 + b_1$, so as to make $a_1 + b_1 = a + b$, we can readily prove, on the foregoing lines, that when $a_1 + a > b_1 + b$, then $a_1 + a > \beta_1 + \beta$. And that $\beta_1 + \beta > \gamma_1 + \gamma$, when $b_1 + b > c_1 + c$, can be proved in like manner.

3. Let $a_1 + a = 2x$, $b_1 + b = 2y$, $c_1 + c = 2z$; and $a_1 - a = 2p$, $b_1 - b = 2q$, $c_1 \sim c = 2r$. Then $x > y > z$, and $p > q > r$; also $x > p$, $y > q$, $z > r$, since the edges all have positive values. Moreover, $a_1 = x + p$, $a = x - p$, $b_1 = y + q$, $b = y - q$, $z + r = c_1$ or c, $z - r = c$ or c_1. But (Euc. I. 20) $b + c > a_1$, $a + c > b_1$, $a + b > c_1$, $b + c_1 > a$, $a + c_1 > b$, and $a + b_1 > c$. Therefore, taking $c_1 > c$, $z > p + q + r$, for $y - q + z - r > x + p$, and even taking $c > c_1$, $z > p + q - r$. But $y - q + z + r > x - p$, therefore $y + 2p > x$. In the same way it can be shown that $z + 2q > y$. We thus discover certain limits. If we take y as a fixed quantity, a_1 and b_1 invariably exceed it, whilst b and

either c or c_1 always fall short of it; the remaining c_1 or c may, however, exceed it, and so may a; moreover, a may exceed b_1, but neither c nor c_1 can do this.

The following orders of magnitude may be doubled by transposing c_1 and c; and, of course, there are transitions between orders, when two magnitudes coincide. Thus a may coincide with b_1, b, c_1, or c; b, again, may coincide with c_1, or c; and c_1 with c,

1. $a_1ab_1bc_1c$	4. $a_1b_1abc_1c$	7. $a_1b_1c_1abc$	10. $a_1b_1c_1cab$	12. $a_1b_1c_1bca$
2. $a_1ab_1c_1bc$	5. $a_1b_1ac_1bc$	8. $a_1b_1c_1acb$	11. $a_1b_1bc_1ca$	13. $a_1b_1c_1cba$.
3. $a_1ab_1c_1cb$	6. $a_1b_1ac_1cb$	9. $a_1b_1c_1bac$		

4. This will be the case when all the dihedral angles are acute, and when the edges follow the same order as the sines of the opposite dihedral angles. Let Z = solid contents of tetrahedron, and Δ_1, Δ_2, Δ_3, Δ_4 = areas of ABC, BCD, ACD, ABD, respectively. Then

$$3Za/2\Delta_3\Delta_4 = \sin\alpha_1,\quad 3Zb/2\Delta_2\Delta_4 = \sin\beta_1,\quad 3Zc/2\Delta_2\Delta_3 = \sin\gamma_1,$$
$$3Za_1/2\Delta_1\Delta_2 = \sin\alpha,\quad 3Zb_1/2\Delta_1\Delta_3 = \sin\beta,\quad 3Zc_1/2\Delta_1\Delta_4 = \sin\gamma.$$

Therefore the required order is followed when $\frac{a_1}{a}, \frac{b_1}{b}, \frac{c_1}{c}, \frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}$ are as $\Delta_1\Delta_2$, $\Delta_1\Delta_3$, $\Delta_1\Delta_4$, $\Delta_3\Delta_4$, $\Delta_2\Delta_4$, $\Delta_2\Delta_3$, which happens when the edges have the order of magnitude, $a_1b_1c_1cba$, given above as the 13th.

INFINITESIMAL OR ZERO? *By the* EDITOR.

"A random point being taken on a given line, what is the chance of its coinciding with a previously assigned point?" That is the question herein discussed, which has arisen incidentally out of Mr. DODGSON's remarks on Question 7695, with criticism thereon by Mr. SIMMONS, given on p. 86 of Volume XLIII., and has been distinctly proposed by him for further discussion, as Question 8200.

I. Mr. DODGSON rejoins as follows:—

"It is surely too late, in A.D. 1885, to seriously discuss the question whether a converging series does or does not reach its limit—in other words, whether an infinitesimal is or is not equal to zero. If the ordinary text-books have not shown Mr. SIMMONS the difference between them, how can I hope to do it? I will, however, try a *reductio ad absurdum*. I present Mr. SIMMONS with a line AB, in which I have selected a certain point C; and I ask him to take a point at random in AB, and to estimate its chance of coinciding with C. He will reply, 'If its chance of falling on one side of C be k, its chance of falling on the other side is, *with perfect accuracy*, $1-k$. Hence its chance of missing C is *absolutely* 1; and its chance of coinciding with it is *absolutely* zero.' But the very same thing is true of *any other* point I might select in AB. Hence the new point *has no chance of falling anywhere!* If Mr. SIMMONS is partial to pitfalls, let me recommend this one to his notice; it is nice soft falling, and not very deep."

II. Mr. SIMMONS, having considered the foregoing reasoning, thus states his objections thereto:—

1. "Mr. DODGSON's anxiety for my comfort is most tender and considerate; but unfortunately in his first sentence he wanders from the question, which (to take a simple case) was whether the difference between two such quantities as 1 and $\cdot\dot{9}$ can be said only *approximately* to equal zero. This use of the italicised word I still maintain to be incorrect.

2. "Here the matter might have ended, but for the remarkable assumption contained in what Mr. DODGSON calls a *reductio ad absurdum*. The argument therein, if it is to have any force at all, plainly assumes that a line may be considered as *wholly made up of points which can all previously be assigned*, a most unmathematical conception! For, even if we take an infinite number of points (which for simplicity of conception, and without affecting the argument, we may consider to be equidistant) on a line, there will still, between every consecutive two, be an infinite number of points remaining. That is to say, the points which have *not* been assigned will always be infinite in number compared with those which *have* been assigned. So that this conception of a line is entirely misleading. It would evidently imply that, when a line is split up into a number of portions sufficiently infinitesimal, the portions either cease to be lines at all, or else they become lines so small that each can contain only a finite number of points; that is to say, the space covered by each bears a finite ratio to the space covered by the points at its extremities!

3. "Now there are two ways in which the chance in question can be proved to be zero. We are of course bound by the definitions given in the text-books of the two words 'point' and 'chance.' For the former, we cannot do better than consult Mr. DODGSON's own edition of Euclid. Are we to infer that he does not consider this to be an 'ordinary' book? For a point is there asserted to have 'position but *no* magnitude.' From the definition of 'chance,' it follows at once that the chance of a point falling on any assigned region or regions of a line is equal to the ratio of the space covered thereby to the space covered by the whole line. Hence, from the two definitions combined, it is clear that the chance of a point falling on a previously assigned point is zero.

4. "Another way of proving it is as follows. Consider AB to be of length unity, and divided into two separate portions at C. If now the length of one portion be k, the length of the other portion will be *with perfect accuracy* $1-k$. Mr. DODGSON will not, I think, challenge this statement. Is he then prepared to deny that it follows, as a necessary consequence, that the two chances are represented quite accurately by k and $1-k$? But, if this be granted, it follows immediately that the chance of coincidence with C is absolutely zero. How Mr. DODGSON, *from the standpoint of the definitions*, can conclude it to be infinitesimal, I am unable to comprehend. I am quite willing to learn, and 'be shown' anything on the subject, but, in the manner given above, he certainly *cannot* 'hope to do it.' As to the 'difference' between infinitesimal and zero, I suppose it is always infinitesimal, while the distinction (is this what Mr. DODGSON really means?) may in certain cases be infinite, especially where ratios are concerned.

5. "The zero result is no doubt remarkable, on the face of it. For does it not appear self-evident, *à priori*, that the new point *must have some* chance of coinciding with C? The apparent contradiction is by no means

easy to explain. Is it that our notion of coincidence implies 'filling the same space,' and that this is inconsistent with the notion of a point as filling no space at all? Or is it that our conception of the possibility of the two points coinciding arises from the possibility of making them coincide by a conscious effort directed to that end, and which may be inconsistent with the 'random' conception? The difficulty, if any exists, is perhaps metaphysical rather than mathematical. At any rate, there seems no escape from the conclusion that, in spite of all preconceived notions to the contrary, the only accurate quantitative expression for the chance of the coincidence of two points, both taken at random on a given line, is absolute and undisguised zero, and nothing but zero."

III. Mr. BIDDLE considers it unwarrantable to define by the term "*absolute* zero" the probability of any event which is not impossible, and draws attention to the essential distinction between such an infinitesimal quantity as $\frac{1}{\infty^\infty}$, and $\frac{0}{1}$. "Probability," he adds, is "clearly a matter of *relativity;* otherwise we should have no right, in the instance given above, to say that the respective chances of the random point falling in AC, BC, were $\frac{AC}{AB}$, $\frac{BC}{AB}$. For, speaking *absolutely*, the length of a line makes no difference to the number of points that can be taken in it. Nor is the case mended by considering how the random point is chosen, viz., by some sort of line cutting AB; for, although speaking *relatively*, the average angular relations of AC, BC to the whole cycle of positions from which such line could be drawn, would give a greater probability to its crossing the longer portion; yet, speaking *absolutely*, we cannot say that more lines can be drawn through the one, than through the other portion. Again, speaking *absolutely*, there are as many points or positions in a square inch as in a square mile; and on that ground we have no right to indicate probabilities by ratios between areas. In fact, the *absolute* theory would do away with Local Probability altogether. In the above discussion, the fact seems to have been lost sight of, that C is not distinct from AC and BC, but belongs to both. The probability in regard to it cannot in reason be treated as if it were even hypothetically additional."

IV. From Mr. MACCOLL we have received the following observations:—

"Whether an *infinitesimal* chance is strictly and logically *zero* depends, of course, upon the definitions we agree to give of the words in italics. But I think it would be convenient if we agreed to define them so as *not* to make them synonymous. Let us take an illustration. A point is taken at random in the circle A, what is the chance that it will also be in the circle B? If the circles touch so as to have one point and one only in common, I should call the chance *infinitesimal;* but if, on the other hand, they neither touch nor intersect, and so have no point in common, I should call the chance *zero*. When conceptions (as in the illustrations just adduced) are different, it is generally convenient to mark this difference by a corresponding difference of language. Whether the same symbol 0 should be used to denote both conceptions, is another question. Practical convenience and the general custom of mathematicians seem in favour of so using it, at least in Probability."

V. Mr. KNOWLES expresses his views on the subject thus:—

"A few years ago I was present at a meeting of the members of the College of Preceptors, when a paper was read by Mr. A. J. ELLIS, F.R.S.,

on 'Incommensurable Quantities,' and one of the subsequent speakers—and Mr. ELLIS appeared to agree with him—stated that an incommensurable quantity might take the form of a recurring decimal. I differed on that occasion, and maintained that a recurring decimal was only an equivalent expression for a vulgar fraction. I am still of that opinion. It is well known that the root of any number never recurs, and if, in $\pi = 3{\cdot}1415 \ldots$ the decimals would only recur, the celebrated problem of squaring the circle would be solved.

"I think, therefore, that Mr. SIMMONS is correct when he states that 1 and $\cdot\dot{9}$ are absolutely equal. These remarks are strictly limited to circulating decimals, but this is important as they crop up in the very elements. With regard to the general question I offer no opinion; it has been a bone of contention ever since the Differential Calculus was discovered, and was the foundation of Bishop BERKELEY's remarks on the illogical methods of the mathematicians of his day."

VI. Lastly, to Mr. SIMMONS' objections, contained above in Section II., Mr. DODGSON sends the following reply:—

1. "I re-affirm that the question whether a converging series does or does not reach its limit *is*, in other words, whether an infinitesimal is or is not equal to zero. *E.g.*—The converging series 2^{-1}, 2^{-2}, &c., 2^{-n}, has, for its limit, unity. Also its sum is $1-2^{-n}$. Hence, if, when n is infinite, the series reaches its limit, the infinitesimal 2^{-n} must be equal to zero.

2. "I never assumed that 'a line may be considered as wholly made up of points which can all previously be assigned,' nor of points of any kind. A point, having no magnitude, can form no portion of a line.

3. "I admit that, if the length of AC, one portion of a line AB, be k, the length of the other portion CB will *with perfect accuracy* be $1-k$. And I *am* 'prepared to deny that the two chances (of a point falling *in* the two portions) are represented quite accurately by k and $1-k$.' For this would omit the 3 chances of its falling at A, at B, and at C. Suppose that, when the point falls at C, it is reckoned as falling in AB, and not in BC. Then, to deal fairly with the two portions, we must exclude A, and make unity represent the chance of the point falling somewhere in the line AB, excluding A, but including B. Then k is the chance of its falling between A and C, or else at C; and $1-k$ the chance of its falling between C and B, or else at B.

4. "I re-affirm, as absolutely axiomatic, that, when an event is *possible*, its chance of happening is *not* zero."

7973. (By Colonel CLARKE, C.B., F.R.S.)—If $\alpha, \beta, \gamma, \delta$ be the angles subtended by the diagonals of a cube at any point of the surface of the inscribed sphere, prove that $\tan^2\alpha+\tan^2\beta+\tan^2\gamma+\tan^2\delta = 8$.

Solution by A. GORDON; Professor MATZ, M.A.; *and others.*

Let A be the point on the sphere (unit radius), and O, its centre, be the origin of coordinates. Then the coordinates of A being $\cos\alpha$, $\cos\beta$, $\cos\gamma$,

the coordinates of the extremities of the diagonals will be $\pm 1, \pm 1, \pm 1$. If the upper-face corners are called 1, 2, 3, 4, and their opposite points $1', 2', 3', 4'$, we have $\quad (11')^2 = (22')^2 = \ldots = 12,$

$(AI)^2 = 4 - 2\cos\alpha - 2\cos\beta - 2\cos\gamma, \;\; (AI')^2 = 4 + 2\cos\alpha + 2\cos\beta + 2\cos\gamma,$

therefore $$\cos I\hat{A}I' = -\frac{12-8}{2\,(AI)\,(AI')},$$

therefore $\quad \tan^2 IAI' = \tan^2\alpha = \frac{1}{4}\left[8 - 8\,(\cos\alpha\cos\beta + \ldots)\right],$

therefore $\quad \tan^2\alpha = 2 - 2\cos\alpha\cos\beta - 2\cos\beta\cos\gamma - 2\cos\gamma\cos\alpha;$

similarly $\tan^2\beta = 2 + 2\cos\alpha\cos\beta - 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha$
$[= \tan^2(2\hat{A}2')],$

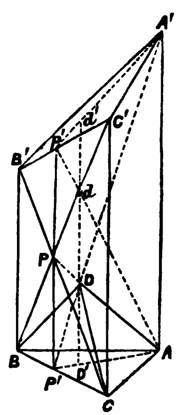
$$\tan^2\gamma = 2 - 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha,$$

$$\tan^2\delta = 2 + 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma - 2\cos\gamma\cos\alpha;$$

hence the result.

8047. (By Professor Malet, F.R.S. Extended in Question 8116.)—If through the vertices of a tetrahedron ABCD any four parallel lines be drawn meeting the opposite faces in A′, B′, C′, D′, prove that the volume of the tetrahedron A′B′C′D′ is three times that of ABCD.

Solution by the Rev. T. C. Simmons, M.A.



Of the four lines, let DD′ be that which falls within the tetrahedron, so that D′ lies within the triangle ABC. The lines AA′, BB′, CC′ will then proceed towards the same direction, and will form a triangular cylinder having ABC for its base. Let the planes ABD. ACD intersect the plane BB′C′C in the lines BC′, CB′, meeting at P, through which draw the parallel p'PP′. Then, since P′P : BB′ = CP : CB′ = C′P : C′B = Pp' : BB′, we get PP′ = Pp'; whence, if the plane AB′C′ meet D′D in d, we obtain from similar triangles in the plane AD′P′$p'd$A, Dd = DD′. That is to say, the intercept made by the plane AB′C′ on D′D produced is equal to DD′, and, as the same thing will follow with regard to the planes BC′A′, CA′B′, it is clear that these three planes all meet D′D in the same point.

Let the same reasoning now be repeated, starting from the end A′B′C′ of the cylinder instead of ABC, and it will be evident that, if D′Dd meet the plane A′B′C′ in d', then Dd = dd'. Hence d'D′ is equal to three times DD′. But the volume of the tetrahedron A′B′C′D′ is one-third of the product of d'D′ by the sectional area of the cylinder, and the volume of the tetrahedron ABCD is one-third of the product of DD′ by the same area. Hence the required result.

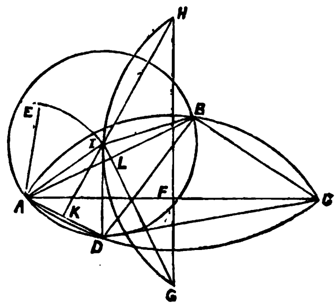
[If straight lines AOA′, BOB′, COC′, DOD′ be drawn, through a point (x, y, z, w), to meet the faces of the fundamental tetrahedron in A′, B′, C′, D′; then at A′, $X = 0, \frac{Y}{y} = \frac{Z}{z} = \frac{W}{w} = \frac{1}{1-x}$, and so for the others, hence the volume of the tetrahedron A′B′C′D′ will be (calling the volume of the fundamental tetrahedron 1)

$$\begin{vmatrix} 0, & \frac{y}{1-x}, & \frac{z}{1-x}, & \frac{w}{1-x} \\ \frac{x}{1-y}, & 0, & \frac{z}{1-y}, & \frac{w}{1-y} \\ \frac{x}{1-z}, & \frac{y}{1-z}, & 0, & \frac{w}{1-z} \\ \frac{x}{1-w}, & \frac{y}{1-w}, & \frac{z}{1-w}, & 0 \end{vmatrix} \text{ or } \frac{3xyzw}{(1-x)(1-y)(1-z)(1-w)}.$$

Now send O to ∞, so that x, y, z, w are infinite (their sum being 1), and the volume of A′B′C′D′ is found to be 3. This assumes that the four parallels are *not* parallel to one, or two, of the faces.]

2235. (By the late Professor TOWNSEND, F.R.S.)—Construct a quadrilateral, having given the two diagonals and the four angles.

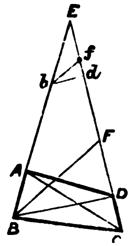
Solutions by (1) D. BIDDLE; (2) Rev. T. C. SIMMONS, M.A.



1. Let AC be the longer of the two diagonals, and G, H the centres from which the loci of B and D are described (Euc. III. 33). Also let AE = the radius of the segment which, with the other diagonal BD as base, contains an angle = A. Upon GH describe the segment GIH, containing the supplement of $\angle$ A, and with A as centre and radius AE describe an arc cutting GIH in I. Join IA, and with centre I and radius IA describe a circle cutting ADC in D, and ABC in B. Join AB, BC, CD, DA, and the required quadrilateral is constructed.

For IA = ID = IB, also HA = HD, and GA = GB; therefore HI produced is perpendicular to AD, and GI is perpendicular to AB. Consequently, $\angle$ GIK = BAD. But GIH = supplement of GIK, and, being contained in the segment GIH, it is also the supplement of $\angle$ A, therefore BAD = $\angle$ A; and, since BD is the base of a segment containing $\angle$ A, of which segment IA (= AE) is radius, BD = the remaining diagonal; and, as B, D, and A are three angles fulfilling the requirements, C must needs be the fourth.

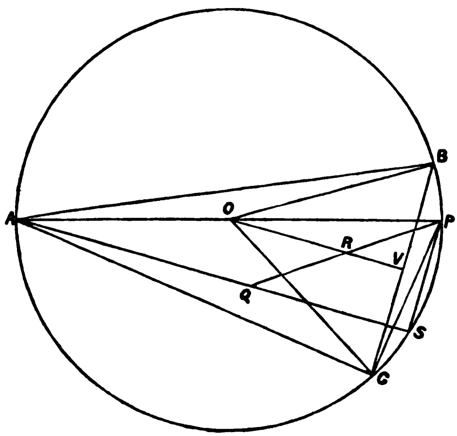
2. *Otherwise*:—Let E be the point of intersection of two opposite sides AB, CD of a quadrilateral whose angles and diagonals are given. Then the triangles EAD, EBC are given in species, whence EB : EC is given, and also ED : EA. Make $\angle$EBF = ECA; then BF : CA and EF : EA are each given, being = EB : EC. Whence BF is given, and also EF : ED.



We have then the very simple problem of determining a point B in one of the sides of a given angle BEC such that BF, BD drawn to the other side may be of given lengths, and EF : ED a given ratio. The solution is as follows. Take on EC any lengths E*f*, E*d* in the given ratio. Let the circular locus of a point whose distances from *f*, *d* are in the ratio of the given lines BF, BD meet EB in *b*, and join *bd*. Take ED : E*d* in the ratio of the given line BD to *bd*. Then D is determined, and the quadrilateral can be at once completed. There will evidently be two solutions corresponding to the two points where the circular locus cuts EB.

8082. (By Professor Hudson, M.A.)—P is any point within the angle A formed by two straight lines AB, AC, to which PB, PC are at right angles, any point Q is taken so that the angle QAB is equal to PAC; prove that the line from the middle point of PQ at right angles to BC bisects BC.

Solution by J. H. Taylor, M.A.; Emily Perrin, B.Sc.; *and others.*



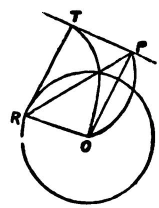
It is plain that a circle will circumscribe ABPC. Produce AQ to meet the circumference in S, and join SP; then $\angle$QAB = PAC (by hypothesis); therefore

$$\angle \text{BAP} = \text{CAS};$$

therefore angles BCP and CPS are equal and SP parallel to CB; also ASP is a right angle, hence a perpendicular from the centre O on BC is parallel to AS and it bisects BC (Euc. iii. 3) and also bisects PQ (Euc. vi. 2).

8092. (By the Rev. J. WHITE, M.A.)—**Show how to draw the normal at any point on the spiral of Archimedes ($r = a\theta$).**

Solution by the PROPOSER.



Draw a perpendicular at the pole to the radius-vector of the point; take on it a length equal to a, and join the point so found with the given point; the joining line is the required normal.

Draw the circle whose radius is a; the spiral of Archimedes is the pedal of the involute of that circle, and intercepts on the tangent to the involute a constant length equal to the radius of the circle (ORTP is obviously a rectangle). The spiral may therefore be considered as generated by a straight line (RT) rolling round the circle, and carrying a line of given length at right angles to it; and the straight line (RP) joining any point on the curve with the centre of instantaneous rotation is the normal.

[If ψ be the angle between the tangent and radius-vector of the spiral,

$$\tan\psi = r\frac{d\theta}{dr}(r = a\theta) = \frac{r}{a} = \cot \mathrm{OPR}\Bigg].$$

NOTE ON QUESTION 7819. *By* Professor NEUBERG.

1. Si D, E, F sont trois points *quelconques* des côtés BC, CA, AB d'un triangle, on a, par une formule *très-connue*,

$$\mathrm{DEF} : \mathrm{ABC} = (\mathrm{BD}\,.\,\mathrm{CE}\,.\,\mathrm{AF} + \mathrm{CD}\,.\,\mathrm{BF}\,.\,\mathrm{AE}) : \mathrm{AB}\,.\,\mathrm{BC}\,.\,\mathrm{CA}.$$

Donc si l'on prend $\mathrm{BD}' = \mathrm{CD}$, $\mathrm{AF}' = \mathrm{AF}$, $\mathrm{CE}' = \mathrm{AE}$, on aura $\mathrm{DEF} = \mathrm{D'E'F'}$.

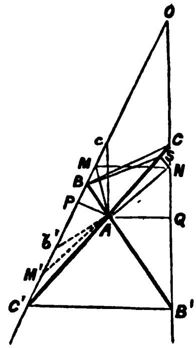
2. Soient D, E, F les pieds des hauteurs. Le point D′ est tel que BC est la sécante minimum menée par D′ dans l'angle BAC (CASEY's *Sequel to Euclid*, p. 39, prop. 18). On conclut de là que le point π, intersection des droites AD′, BE′, CF′, jouit de la propriété que les parallèles aux côtés de ABC menées par π et limitées aux angles opposés sont des sécantes minimum passant par π. π est le centre de gravité de trois masses égales à cot A, cot B, cot C, appliquées en A, B, C. Les distances de π aux côtés de ABC sont égales à AD cot A tan α, BE cot B tan α, CF cot C tan α, α étant l'angle de Brocard. Les parallèles minimum menées par π ont pour valeurs
$$2\mathrm{R}\,\frac{\sin^2 \mathrm{A}}{\sin \mathrm{B} \sin \mathrm{C}} \tan\alpha, \text{ etc.};$$
elles sont donc proportionnelles aux cubes des côtés.

3. Soit A′B′C′ le triangle que forment les parallèles à BC, CA, AB, menées par A, B, C; D′, E′, F′ sont les milieux des hauteurs de A′B′C′; donc π est le centre des symédianes de A′B′C′.

[Solutions of Question 7819 are given on pp. 64—65 of Vol. XLII.]

8052. (By the EDITOR.)—A given angle turns round a fixed point A and cuts the sides of a given fixed angle O in B, B′, and C, C′; prove, both by elementary geometry and by trigonometry, that each of the triangles ABC, AB′C′ is a minimum when the inclination of AB or AC to the sides of the angle O is half the sum of the two given angles.

Solution by the Rev. T. C. SIMMONS, M.A.



Draw AP, AQ perpendiculars on the sides of the given angle, and make $\angle AcP = ACQ$, and $\angle AMP$ = half ($\angle BAC + \angle BOC$). Then we have

$$2\angle AMP = ACQ + ABP = AcP + ABP,$$

or $$\angle AMP - AcP = ABP - AMP;$$

i.e., AM always bisects $\angle BAc$. Now Ac is always to AC in the fixed ratio of AP to AQ. Hence the area of $\triangle ABC$, which varies as AB, AC, varies also as $AB \cdot Ac$; that is (by Euc. vi. B), as $AM^2 + BM \cdot Mc$. So that the area will be a minimum when $BM \cdot Mc$ vanishes, *i.e.*, when B coincides with M.

Again, taking $\angle Ab'P = \angle AB'Q$, and $\angle AM'P$ = half of ($\angle AB'Q + AC'P$), *i.e.*, half of ($\angle BAC - BOC$), it will be seen in the same way that $\triangle AB'C'$ is a minimum when C′ coincides with M′. Moreover,

as $$\angle AM'P + AMP = \tfrac{1}{2}(BAC + BOC + BAC - BOC) = \angle BAC,$$

it is plain that, when B coincides with M, C′ will coincide with M′; so that both triangles are a minimum at the same time.

An easy trigonometrical proof may also be given as follows:—Let $\angle PAM = \alpha$, $\angle BAM$ or $cAM = \theta$, then $AB \cdot Ac$ will be a minimum when $AP \sec(\theta - \alpha) \cdot AP \sec(\theta + \alpha)$ is a minimum, that is, when $\cos(\theta - \alpha)\cos(\theta + \alpha)$ or $\cos 2\alpha + \cos 2\theta$ is a maximum; that is, when $\theta = 0$.

[If AMN be the triangle that makes $\angle AMP = ANQ = \tfrac{1}{2}(MON + MAN)$, and we make $\angle ANS = ABM$, and join BS, we shall have $\angle ANC = AMO > ABM$, so that S, in AC, lies between A and C; also the triangles AMB, ANS are similar, therefore $AB : AM = AN : AS$; thus it follows that $\triangle AMN = ABS$, and is therefore less than any other such triangle as ABC; and the like holds good for all other positions.]

8106. (By W. J. MCCLELLAND, B.A.)—Having given the base c and $l \cot A + m \cot B$ of a spherical triangle; find the locus of the vertex.

Solution by D. EDWARDES; *the* PROPOSER; *and others.*

If (x, y) be the coordinates of the vertex, the mid-point of the base being the origin, we have from the two right angled-triangles

$$\sin(x + \tfrac{1}{2}c) = \cot A \tan y, \quad \sin(\tfrac{1}{2}c - x) = \cot B \tan y;$$

the equation of the locus is therefore

$$n \tan y = l \sin(\tfrac{1}{2}c + x) + m \sin(\tfrac{1}{2}c - x) \quad \text{or} \quad n \tan y = \mathrm{A} \cos(x - x'),$$

where $$\mathrm{A}^2 = l^2 + m^2 - 2lm \cos c, \quad \tan x' = \frac{l-m}{l+m} \cot \frac{c}{2}.$$

The locus is, therefore, a great circle, and, if $l = m$, the pole lies on the great circle bisecting the base at right angles.

[If a point X be taken on the base AB and joined to the vertex C, we have the general relation $\cot \mathrm{A} \sin \mathrm{BX} + \cot \mathrm{B} \sin \mathrm{AX} + \cot \theta \sin \mathrm{C}$, where θ is the angle between CX and AB; hence, if $\sin \mathrm{AX} : \sin \mathrm{BX} = m : l$, we have the first two terms in our equation, and consequently θ; thus the locus of the vertex is a great circle cutting the base at a fixed point X and at a given angle θ.]

8083. (By Professor COCHEZ.)—Trouver une courbe dont le rayon de courbure est proportionnel au cube de la normale.

Solution by Rev. T. C. SIMMONS, M.A.; EMILY PERRIN, B.Sc.; *and others.*

We have $$\left(\frac{ds}{dx}\right)^3 \Big/ \frac{d^2y}{dx^2} \propto y^3 \left(\frac{ds}{dx}\right)^3, \quad \text{or} \quad \frac{d^2y}{dx^2} \propto \frac{1}{y^3} = \pm \frac{b^4}{a^2y^3}, \text{ (say)};$$

therefore $$\left(\frac{dy}{dx}\right)^2 = \pm \frac{b^4}{a^2y^2} + \text{const.} = \pm \frac{b^4}{a^2y^2} \pm \frac{b^2}{a^2} \text{ (say)};$$

or $$\frac{ay\,dy}{(\pm b^2 \pm y^2)^{\frac{1}{2}}} = b\,dx, \quad \text{whence} \quad a^2(\pm b^2 \pm y^2) = b^2(x+h)^2,$$

or $$\frac{(x+h)^2}{a^2} \pm \frac{y^2}{b^2} = \pm 1,$$

denoting a central conic, one of whose principal axes coincides with the axis of x. Also by taking the left-hand sign negative, the right-hand sign positive, and putting $h = a$, $b^2 = ma$, we get $y^2 = mx^2a^{-1} + 2mx$, which, when a is infinite, denotes a parabola whose principal diameter is the axis of x. Thus all the three conics are included, which is in accordance with well-known results.

8113. (By W. J. C. SHARP, M.A.)—Show that the series

$$1 + n\cos\theta + \frac{n(n-1)}{1\,.\,2}\cos 2\theta + \&\text{c.} = (2\cos\tfrac{1}{2}\theta)^n \cos\tfrac{1}{2}n\theta = 0 \text{ if } \theta = \frac{\pi}{n} \text{ or } n\pi,$$

$$n\sin\theta + \frac{n(n-1)}{1\,.\,2}\sin 2\theta + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\sin 3\theta + \&\text{c.} = (2\cos\tfrac{1}{2}\theta)^n \sin\tfrac{1}{2}n\theta = 0$$

if $$\theta = \frac{2\pi}{n} \text{ or } n\pi.$$

Solution by the Rev. J. J. Milne, M.A.

Let $C = 1 + n\cos\theta + \ldots$, and $S = n\sin\theta + \ldots$, therefore

$$C + Si = 1 + ne^{\theta i} + \ldots = (1 + e^{\theta i})^n = e^{n\frac{1}{2}\theta i}\left(e^{\frac{1}{2}\theta i} + e^{-\frac{1}{2}\theta}{}_i\right)^n$$
$$= (2\cos\tfrac{1}{2}\theta)^n(\cos\tfrac{1}{2}n\theta + i\sin\tfrac{1}{2}n\theta);$$

therefore, equating real and unreal parts, we have the result.

[The two series are, states Prfoessor Macfarlane, by "plane algebra" together

$$= 1 + n(\cos\theta + i\sin\theta) + \frac{n(n-1)}{1\,.\,2}(\cos 2\theta + i\sin 2\theta) + = (1\cdot 0 + 1\cdot\theta)^n,$$

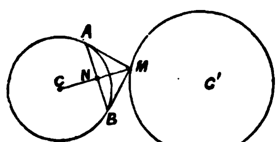
where the first symbol denotes the length, and the second symbol the angle of the vector,

$$= (2\cos\tfrac{1}{2}\theta\,.\,\tfrac{1}{2}\theta)^n = (2\cos\tfrac{1}{2}\theta)^n\,.\,\tfrac{1}{2}(n\theta) = (2\cos\tfrac{1}{2}\theta)^n\left\{\cos\tfrac{1}{2}(n\theta) + i\sin\tfrac{1}{2}(n\theta)\right\}].$$

8112 & 8147. (By R. Knowles, B.A., and Rev. J. J. Milne, M.A.) —Given two circles which do not intersect; from any point on the one tangents are drawn to the other: prove that the envelope of the chord of contact and the locus of its middle point are respectively a conic and its auxiliary circle.

Solutions by (1) Dr. Curtis; (2) Prof. Neuberg *and* Rev. J. J. Milne.

1. If from a point on *any curve*, B-tangents, *whether real or imaginary*, be drawn to a circle A whose centre is at O, the envelope of the chord of contact will be the A-polar of B, and the locus of the foot of the perpendicular let fall from O on this chord of contact, the middle point of the chord if the tangents are real, will be the O pedal of this polar curve. If the curve B be a circle, the polar will be a conic having O for a focus, and the pedal curve, consequently, the circle which touches it at the extremities of its axis major.



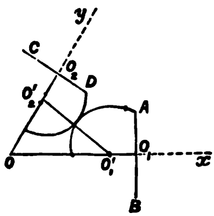
2. Soient C, C′ deux cercles *quelconques;* MA et MB les tangentes menées d'un point quelconque M de C′ au cercle C. La corde des contacts AB a son milieu M sur CM. Comme $(CA)^2 = CN\,.\,CM$, le point N décrit une courbe inverse du cercle M, c'est-à-dire un cercle C″.

L'enveloppe de AB est une conique Σ ayant pour foyer C et pour cercle décrit sur l'axe focal comme diamètre le cercle C″. Σ et C″ ont donc un double contact aux sommets de Σ.

[If the circle intersect, as in III., and for "chord of contact" we read "polar," as in solution (1), the conic will be an ellipse or hyperbola, according as the centre of C is within or without C′. If C be on C′, the locus of the foot of the perpendicular from C on the polar is a straight line and the envelope is a parabola. Thus the envelope is an ellipse, hyperbola, or parabola, according as the centre of C is within, without, or upon C′.]

8090. (By Colonel CLARKE, C.B., F.R.S.)—A circle passing through two fixed points touches a circle passing through two other fixed points; determine the locus of the point of contact.

Solution by A. GORDON.



Let AB, CD be the pairs of given points, and (ξ, η) the coordinates of the point of contact; and let $CD = 2b$, $AB = 2a$, $\mu = OO_2$, $\lambda = OO_1$ (which are known), $f = OO_2'$, $g = OO_1'$, r_2, r_1 the radii, then

$$(\mu-f)^2+b^2 = r_2^2,\quad f^2+g^2-2gf\cos\omega = (r_1+r_2)^2,$$

$$(\lambda-g)^2+a^2 = r_1^2,\quad \xi = \frac{r_2}{r_1+r_2}g,\quad \eta = \frac{r_1}{r_1+r_2}f.$$

Between these five equations it is necessary to eliminate r_1, r_2, f, g; hence

$$\text{(i.)}\ \xi f+g\eta = gf,\quad \text{(ii.)}\ \frac{f^2+g^2-2fg\cos\omega}{(\lambda-g)^2+a^2} = \frac{f^2}{\eta^2},\quad f = \frac{g\eta}{g-\xi},\ \text{from (i.)}$$

$$\frac{\left(\mu-\dfrac{g\eta}{g-\xi}\right)^2+b^2}{(\lambda-g)^2+a^2} = \frac{\xi^2 f^2}{\eta^2 g^2} = \frac{\xi^2}{(g-\xi)^2};$$

therefore $\quad [\mu(g-\xi)-g\eta]^2+(g-\xi)^2b^2 = \xi^2[(\lambda-g)^2+a^2]$ ....(iii.),

but $\quad g = \dfrac{\eta^2+\xi^2+2\eta\xi\cos\omega-\lambda^2-a^2}{2(\xi+\eta\cos\omega-\lambda)}$ by substituting in (ii.) for f;

hence, substituting for g and reducing in (iii.), we have

$$[\mu(\eta^2-\xi^2-\lambda^2-a^2+2\lambda\xi)-\eta(\eta^2+\xi^2+2\eta\xi\cos\omega-\lambda^2-a^2)]^2$$
$$+b^2[\eta^2-\xi^2-\lambda^2-a^2+2\lambda\xi]^2$$
$$= \xi^2[2\xi\lambda+2\eta\lambda\cos\omega-\lambda^2+a^2-2\eta\xi\cos\omega-\eta^2-\xi^2]^2+4\xi^2a^2(\xi+\eta\cos\omega-\lambda]^2.$$

This equation is in general of the 6th degree.

[The factor $(\lambda-\xi)^2+a^2-\eta^2$ will divide out, thus reducing the equation to the 4th degree. If we write u, v for the reciprocals of r_1, r_2, and then divide the equation $f^2+g^2-2gf\cos\omega = (r_1+r_2)^2$ by $(r_1+r_2)^2$, and the equations $(\mu-f)^2+b^2 = r_2^2$, $(\lambda-g)^2+a^2 = r_1^2$, by r_2^2 and r_1^2 respectively, we shall get these three equations:—

$$\eta^2u^2 + \xi^2v^2-2\eta\xi\cos\omega\, uv = 1 \text{ ....(1)},$$

$$\eta^2u^2+[(\mu-\eta)^2+b^2]v^2-2\eta(\mu-\eta)uv = 1 \text{ ....(2)},$$

$$[(\lambda-\xi)^2+a^2]u^2+\xi^2v^2-2\xi(\lambda-\xi)uv = 1 \text{ ....(3)},$$

(2) − (1) gives $[(\mu-\eta)^2+b^2-\xi^2]v^2-2\eta[\mu-\eta-\xi\cos\omega]uv = 0$,

(3) − (1) gives $[(\lambda-\xi)^2+a^2-\eta^2]u^2-2\xi[\lambda-\xi-\eta\cos\omega]uv = 0$;

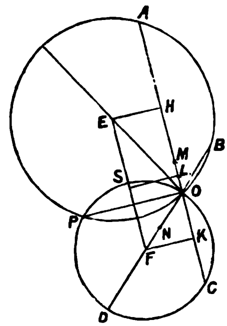
eliminating $u : v$, the result is

$$[(\mu-\eta)^2+b^2-\xi^2][(\lambda-\xi)^2+a^2-\eta^2]-4\xi\eta[\mu-\eta-\xi\cos\omega][\lambda-\xi-\eta\cos\omega] = 0,$$

a symmetrical equation of the 4th order.]

8111. (By E. Rutter.)—ABCD is a quadrilateral; M, N the mid-points of the diagonals AC, BD, whose point of intersection is O; circles OAB, OCD intersect in P, and circles OBC, ODA in Q: prove that the points M, N, O, P, Q lie on the circumference of a circle.

Solutions by (1) Emily Perrin, B.Sc.; (2) Professor Neuberg.



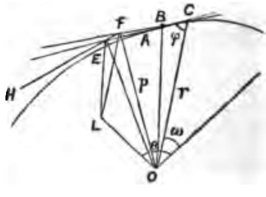
1. Let E, F be the centres of the two circles; then the centre of any circle through O, P lies in EF. Draw EH, FK perpendicular to AC, and LS through L the mid-point of OM; then, since HN = NK, therefore S bisects EF; similarly the centre of the circle through OPN lies at S, therefore O, P, N, M are concyclic, as are, similarly, O, Q, N, M; hence the five points O, P, Q, N, M are concyclic.

[Of course, the four centres of the circles OAB, OBC, OCD, ODA form the angular points of a parallelogram.]

2. Menons les diamètres OE, OF, OG, OH des cercles OAB, OBC, OCD, ODA. La droite EF passe par B, et est perpendiculaire à BD, EG passe par P, etc. La figure EFGH est un parallélogramme dont les diagonales EG, FH se coupent en leur milieu commun R. La circonférence qui a pour diamètre OR passe par les sommets des angles droits OPR, OQR, OMR, ONR.

8110. (By the Editor.)—If the pedal of any closed oval curve whose area is A be taken with respect to an internal point, and through each point of the pedal a straight line be drawn making an angle α with the radius vector to that point, prove that the area (A′) of the curve enveloped by these lines, is given by the equation $A' = A\sin^2\alpha$.

Solutions by (1) Dr. Curtis; (2) Professor Neuberg.



1. If O be the origin, A, B, C three consecutive points on the oval curve, OE, OF ($=p$) perpendiculars on the tangents AB, BC, and EL, FL the lines which make, with OE, OF, angles equal α; then L is a point on the curve-envelope of the line referred to in the question. Denoting by (r, ω), (r, θ), the polar coordinates of C and L respectively, and by ϕ the angle OCB, as quadrilaterals OFEL and OCFE

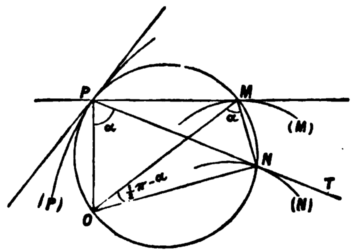
are each circumscribable by a circle, $\angle$ OLE $= \pi - \angle$ OFE $= \pi - \angle$ OOB, therefore $\rho = p \sin\alpha \operatorname{cosec}\phi = r\sin\alpha$ ....(1),

and $\theta - \omega = \angle \text{LOC} = \angle \text{LOE} + \angle \text{EOC} = \angle \text{LFE} + \angle \text{EOC}$

$= \angle \text{OFH} - \alpha + \angle \text{EOC} = \phi + \angle \text{EOC} - \alpha = \frac{1}{2}\pi - \alpha,$

therefore $d\theta - d\omega = 0$, or $d\omega = d\theta$, and therefore, from (1), $\rho^2 d\omega = \sin^2\alpha\, r^2\, d\theta$, and, integrating, $A' = A\sin^2\alpha$.

2. Soient (N) une courbe quelconque, OP une oblique menée d'un point fixe O sous l'angle constant OPN vers la tangente NT. La tangente au point P de l'α-pédale (P) touche le cercle ONP.



Réciproquement, si l'on considère une courbe (P) comme étant la pédale d'une courbe (M) ou l'α-pédale d'une courbe (N), on construira le cercle qui passe par O, P et touche la courbe (P) en P; ce cercle coupera la droite PM perpendiculaire à OP en un point M de la pédale, et coupera la droite PN faisant avec OP l'angle OPN $= \alpha$, en un point N de l'α-pédale.

Il résulte de là que l'angle OMN $= \alpha$, MON $= \frac{1}{2}\pi - \alpha$. Donc les courbes (M) et (N) sont semblables: les rayons vecteurs OM, ON font l'angle constant $\frac{1}{2}\pi - \alpha$, et sont dans le rapport constant $1 : \sin\alpha$. Les aires décrites par OM et ON sont dans le rapport $1 : \sin^2\alpha$.

Les lignes décrites par M et N sont proportionelles à 1 et $\sin\alpha$.

7854. (By J. Brill, B.A.)—If a, b, c be the sides of a triangle inscribed in a parabola, and P, Q, R the focal chords parallel to them; prove that

$$\frac{a}{\sqrt{P}} + \frac{b}{\sqrt{Q}} + \frac{c}{\sqrt{R}} = 0.$$

Solution by R. Knowles, B.A.; W. J. Greenstreet, B.A.; *and others.*

If (h, k) be the coordinates of the pole of a chord of the parabola $y^2 = 4a'x$, and $y_1 y_2 y_3$ the ordinates of the vertices of the triangle, we have

$$a = \frac{1}{a}(k^2 + 4a'^2)^{\frac{1}{2}}(k^2 - 4a'h)^{\frac{1}{2}}; \text{ and if } h = -a',\ P = \frac{1}{a'}(k^2 + 4a'^2);$$

therefore $$\frac{a}{\sqrt{P}} = a'^{-\frac{1}{2}}(k^2 - 4ah)^{\frac{1}{2}} = \frac{a'^{-\frac{1}{2}}}{2}(y_1 - y_2),$$

and similarly of the other ratios; hence

$$\frac{a}{\sqrt{P}} + \frac{b}{\sqrt{Q}} + \frac{c}{\sqrt{R}} = \frac{a'^{\frac{1}{2}}}{2}(y_1 - y_2 + y_2 - y_3 + y_3 - y_1) = 0.$$

7757. (By W. J. GREENSTREET, B.A.)—Prove that the sum of the series (when congruent) $x - \frac{x^7}{7!} + \frac{x^{13}}{13!} - \frac{x^{19}}{19!} + \ldots + (-1)^r \frac{x^{6r+1}}{6r+1!} + \ldots$

$$\frac{\sin x}{3} + \frac{1}{3} e^{\frac{1}{2}\sqrt{3}x} \sin\left(\frac{x}{2} + \frac{\pi}{3}\right) + \frac{1}{3} e^{-\frac{1}{2}x\sqrt{3}} \sin\left(\frac{x}{2} - \frac{\pi}{3}\right).$$

Solution by R. KNOWLES, B.A.; G. G. STORR, B.A.; *and others.*

This kind of series is dealt with in DE MORGAN's *Calculus*, p. 319, the solution depending on the 6 roots of -1. Suppose there are n roots of the form $\alpha \pm \beta\sqrt{-1}$, the solution will consist of the sum of terms of the form

$$\left\{\frac{e^{(\alpha+\beta\sqrt{-1})x}}{\alpha+\beta\sqrt{-1}} + \frac{e^{(\alpha-\beta\sqrt{-1})x}}{\alpha-\beta\sqrt{-1}}\right\} \div n;$$

which becomes $\qquad \frac{2e^{\alpha x}}{n}(\alpha \cos \beta x + \beta \sin \beta x)$ ....(1).

The values of α and β are respectively $(0\ 1)$ $\left(\frac{\sqrt{3}}{2}\ \frac{1}{2}\right)$ $\left(-\frac{\sqrt{3}}{2}\ \frac{1}{2}\right)$; substituting in (1), we have for the sum of the series the given expression.

8061. (By Rev. T. R. TERRY, M.A.)—A thin smooth spherical shell of radius a and mass M rests on a horizontal plane, and a particle of mass m is placed inside at the extremity of a horizontal diameter of the shell, and then let go; find (1) the velocity of the particle relative to the shell in any position; also (2), if V be the velocity of the shell when the particle reaches the lowest point, prove that $M(M+m)V^2 = 2m^2ag$.

Solution by D. EDWARDES; EMILY PERRIN; *and others.*

Let u be the velocity of the sphere at time t, θ the angle turned through by the radius drawn to the particle. The velocity components of the particle are then $u - a\dot{\theta}\sin\theta$, $-a\dot{\theta}\cos\theta$, and therefore

$$Mu + m(u - a\dot{\theta}\sin\theta) = 0 \quad\text{..............................(1).}$$

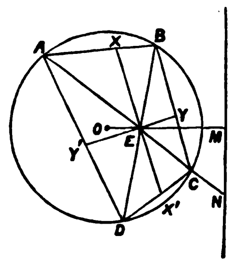
The equation of energy is

$$Mu^2 + m(u^2 - 2au\,\dot{\theta}\sin\theta + a^2\dot{\theta}^2) = 2mg\,a\sin\theta \quad\text{..............(2)};$$

whence, eliminating u, $\left(1 - \frac{m\sin^2\theta}{M+m}\right)\dot{\theta}^2 = \frac{2g}{a}\sin\theta$, which determines the relative velocity of the particle. When $\theta = \frac{1}{2}\pi$, $M\dot{\theta}^2 = \frac{2g}{a}(M+m)$; whence, by (1), $\qquad V^2M(M+m) = 2m^2ag.$

7679. (By W. J. McClelland, B.A.)—For any quadrilateral ABCD inscribed in a circle whose diagonals intersect in a limiting point; if the bisectors of the angles between the diagonals meet the four sides respectively in points X, Y, X′, and Y′, prove that the products of the distances of the points A, B, C, D and X, Y, X′, Y′ from the radical axis are equal to one another.

Solution by B. Hanumanta Rau, B.A.; Belle Easton; *and others.*



Let the distance of A, B, C, D, &c. from the radical axis be $a, b, c, \ldots$ Then, since

$$NC : NE = NE : NA,$$

$$ac = EM^2 \text{ and } bd = EM^2, \quad \therefore\ abcd = EM^4.$$

Again (by Question 7765),

$$2OE \,.\, x = AX \,.\, XB + XE^2 = AE \,.\, EB$$

$$(\because\ EX \text{ bisects } \angle AEB),$$

$$\therefore\ 4OE^2 \,.\, xx' = AE \,.\, EB \,.\, CE \,.\, ED$$

$$= (OB^2 - OE^2)^2 = (2 \,.\, OE \,.\, EM)^2,$$

$$\therefore\ xx' = EM^2; \text{ similarly } yy' = EM^2,$$

or $$xx'yy' = EM^4 = abcd.$$

In fact, $$ac = bd = xx' = yy' = EM^2.$$

7859 & 7950. (7859).—(By Belle Easton.)—A square is divided into 16 equal squares by vertical and horizontal lines; find in how many ways 4 of these can be painted white, 4 black, 4 red, and 4 blue, without repeating the same colour in the same vertical or horizontal row.

(7950.) (By B. Hanumanta Rau, M.A.)—Find in how many ways a pack of 52 cards can be distributed among four persons, so that each may have ace, king, queen, and knave, but all of different suits.

Solution by the Rev. T. C. Simmons, M.A.; G. G. Storr, B.A.; *and others.*

a	d	b
b		
c		
d		

(7859.) Denoting the colours by a, b, c, d, let $abcd$, $adcb$ be one of the $4\,! \times 3\,!$ ways of arranging the first column and first row. Then in the third square of the second row we can have either a or c or d, and the possible arrangements of this row will consequently be $bcad$, $bacd$, $bcda$, the first of which evidently gives two ways, and each of the others only one way, of completing the whole square.

Hence the total possible number of arrangements

$$= 4\,! \times 3\,! \times 4 = [4\,!]^2 = 576.$$

(7950.) The number of ways in which the court cards can be distributed is seen from the preceding to be $[4!]^2$. For we can clearly take the successive columns to denote the hands of the players, the rows the suits, and a, b, c, d the ace, king, queen, knave respectively. The number of ways in which the remaining cards can then be distributed

$$= \frac{(36)!}{9!\,(27)!} \cdot \frac{(27)!}{9!\,(18)!} \cdot \frac{(18)!}{9!\,9!} = \frac{(36)!}{[9!]^4}.$$

Hence total possible number of arrangements $= \dfrac{(36)!\,[4!]^2}{[9!]^4}$.

7568. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Having given $\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0$, prove each of the following equations:—

$$[\Sigma(\cos\alpha)-\cos(\alpha+\beta+\gamma)]^2 / \cos\alpha\cos\beta\cos(\alpha+\beta+\gamma)$$

$$= [\Sigma(\sin\alpha)+\sin(\alpha+\beta+\gamma)]^2 / -\sin\alpha\sin\beta\sin\gamma\sin(\alpha+\beta+\gamma) = 4,$$

$$[\Sigma(\cos\alpha)-\cos(\alpha+\beta+\gamma)]\,[\Sigma(\sec\alpha)-\sec(\alpha+\beta+\gamma)] = 4,$$

$$[\Sigma(\sin\alpha)+\sin(\alpha+\beta+\gamma)]\,[\Sigma(\operatorname{cosec}\alpha)+\operatorname{cosec}(\alpha+\beta+\gamma)] = 4;$$

$$\left(\frac{\cos\alpha\cos\beta\cos\gamma}{\cos(\alpha+\beta+\gamma)}\right)^{\frac{1}{2}} + \left(\frac{\sin\alpha\sin\beta\sin\gamma}{-\sin(\alpha+\beta+\gamma)}\right)^{\frac{1}{2}} = 1,$$

$$\left(\frac{\cos\beta\cos\gamma\cos(\alpha+\beta+\gamma)}{\cos\alpha}\right)^{\frac{1}{2}} + \left(\frac{\sin\beta\sin\gamma\sin(\alpha+\beta+\gamma)}{-\sin\alpha}\right)^{\frac{1}{2}} = 1,$$

$$\left(\frac{\cos\gamma\cos\alpha\cos(\alpha+\beta+\gamma)}{\cos\beta}\right)^{\frac{1}{2}} + \left(\frac{\sin\gamma\sin\alpha\sin(\alpha+\beta+\gamma)}{-\sin\beta}\right)^{\frac{1}{2}} = 1,$$

$$\left(\frac{\cos\alpha\cos\beta\cos(\alpha+\beta+\gamma)}{\cos\gamma}\right)^{\frac{1}{2}} + \left(\frac{\sin\alpha\sin\beta\sin(\alpha+\beta+\gamma)}{-\sin\gamma}\right)^{\frac{1}{2}} = 1.$$

Hence, if $u=0$ represent any one of these equations (rationalized if necessary, and cleared of fractions), $\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)$ must be a factor of u. Write each equation in a form exhibiting this factor and the remaining factors of this equation.

Solution by B. HANUMANTA RAU, B.A.; Professor N. SARKAR, M.A.; *and others.*

Putting x for $\alpha+\beta+\gamma$, we have $\sin(x-\alpha)+\sin(x-\beta)+\sin(x-\gamma)=0$,

or $\dfrac{\Sigma(\sin\alpha)}{\sin x} = \dfrac{\Sigma(\cos\alpha)}{\cos x} = [3+2\cos(\beta-\gamma)+2\cos(\gamma-\alpha)+2\cos(\alpha-\beta)]^{\frac{1}{2}}$ ....(1),

$= \cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)$ ....(2).

Again, by hypothesis,

$$\sin\tfrac{1}{2}(\alpha+\beta+2\gamma)\cos\tfrac{1}{2}(\alpha-\beta)+\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha+\beta)=0,$$

or $$\sin\tfrac{1}{2}(x+\gamma)\cos\tfrac{1}{2}(\alpha-\beta)+\sin\tfrac{1}{2}(x-\gamma)\cos\tfrac{1}{2}(\alpha+\beta)=0,$$

or $$\sin\tfrac{1}{2}x\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma+\cos\tfrac{1}{2}x\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma=0,$$

or $$\frac{-\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma}{\sin\frac{1}{2}x}=\frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma}{\cos\frac{1}{2}x}=\tfrac{1}{2}\left(\frac{-\sin\alpha\sin\beta\sin\gamma}{\sin x}\right)^{\frac{1}{2}}$$

$$=\tfrac{1}{2}\left(\frac{1+\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta}{\cos x}\right)^{\frac{1}{2}}$$

$$=\tfrac{1}{2}\left(\frac{\Sigma(\cos\alpha)+\cos\alpha\cos\beta\cos\gamma}{\cos x}\right)^{\frac{1}{2}}\ \text{..........................(3)}.$$

From (1) and (2), we have

$$\left(\frac{\Sigma(\sin\alpha)}{\sin x}\right)^2+2\frac{\Sigma(\sin\alpha)}{\sin x}=\left(\frac{\Sigma(\cos\alpha)}{\cos x}\right)^2+2\frac{\Sigma(\cos\alpha)}{\cos x}$$

$$=3+4(\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta)$$

$$=-1-4\frac{\sin\alpha\sin\beta\sin\gamma}{\sin x}=4\frac{\Sigma\cos\alpha+\cos\alpha\cos\beta\cos\gamma}{\cos x},\ \text{by (3)},$$

therefore $$[\Sigma(\sin\alpha)+\sin x]^2=-4\sin\alpha\sin\beta\sin\gamma\sin x\ \text{...............(4)},$$

and $$[\Sigma(\cos\alpha)-\cos x]^2=\ \ 4\cos\alpha\cos\beta\cos\gamma\cos x\ \text{...............(5)}.$$

From (3), $$\frac{\Sigma(\cos\alpha)-\cos x}{\cos\alpha\cos\beta\cos\gamma\cos x}=\Sigma(\sec\alpha)-\sec x;$$

therefore, by (5), $[\Sigma(\cos\alpha)-\cos x][\Sigma(\sec\alpha)-\sec x]=4.$

Again, from (3) and (2), we obtain

$$1+\sin\beta\sin\gamma+\sin\gamma\sin\alpha+\sin\alpha\sin\beta=-\frac{\sin\alpha\sin\beta\sin\gamma}{\sin x}-\frac{\Sigma\sin\alpha}{\sin x},$$

or $$\Sigma\operatorname{cosec}\alpha+\operatorname{cosec}x=\frac{\Sigma\sin\alpha+\sin x}{-\sin\alpha\sin\beta\sin\gamma\sin x};$$

therefore, by (4), $[\Sigma\sin\alpha+\sin x][\Sigma\operatorname{cosec}\alpha+\operatorname{cosec}x]=4.$

From (4) and (5),

$$\left(\frac{\sin\alpha\sin\beta\sin\gamma}{-\sin x}\right)^{\frac{1}{2}}+\left(\frac{\cos\alpha\cos\beta\cos\gamma}{\cos x}\right)^{\frac{1}{2}}$$

$$=\tfrac{1}{2}\left(\frac{\Sigma\sin\alpha}{\sin x}+1\right)\pm\tfrac{1}{2}\left(\frac{\Sigma\cos\alpha}{\cos x}-1\right)=1,$$

the lower sign being taken.

Again, $$\left(\frac{\sin\beta\sin\gamma\sin x}{-\sin\alpha}\right)^{\frac{1}{2}}+\left(\frac{\cos\beta\cos\gamma\cos x}{\cos\alpha}\right)^{\frac{1}{2}}$$

$$\tfrac{1}{2}\frac{\Sigma\sin\alpha+\sin x}{\sin\alpha}\pm\tfrac{1}{2}\frac{\Sigma\cos\alpha-\cos x}{\cos\alpha}$$

$$=1+\tfrac{1}{2}\frac{\sin(\alpha+\beta)+\sin(\alpha+\gamma)+\sin(x-\alpha)}{\sin\alpha\cos\alpha}=1.$$

Similarly for the next two equations.

7788. (By Âsútosh Mukhopâdhyây, B.A., F.R.A.S.)—A cavity is scooped out of a solid sphere of given substance; all that is known is that it is one of the regular solids concentric with the sphere. Determine *mechanically, without opening or penetrating the sphere*, the form and dimensions of the cavity.

Solution by D. Biddle.

The cubic space occupied by the cavity might easily be found by comparing the weights of a whole sphere and of that scooped out. Then the position of each angle of the cavity could be determined by experiment, since planes tangential to an imaginary sphere of size equal to the cavity and concentric with it would cut off from the original sphere segments differing in weight according as they were parallel to the sides of the cavity or perpendicular to a line joining the centre with one of the angles.

If the hollow sphere, therefore, were placed on a horizontal surface, or (still better) floated in a liquid, an angle of the cavity would, when at rest, be uppermost; and by rotating the sphere in various directions, and noting where the equilibrium was more stable, and where less, the position of all the angles of the cavity could be indicated. We should thus arrive at their number, and the rest would be easy. The sphere should be floated in a liquid allowing it to sink half-way, and then more liquid of a lighter kind should be slowly added allowing it to sink still further, until the position of equilibrium under the particular conditions became clearly indicated. One or other angle would remain above the level of the liquid until the whole hollow space was submerged.

3886. (By the Editor.)—A. pays B. £5 for the following privilege:—Three dice are to be thrown, and A. is to receive from B. in shillings the square of the sum of the numbers turned up if they contain a doublet, and the cube if they form a triplet; but, in any other case, he is to give B as many shillings as the sum of the numbers turned up. Which of them is the more likely to gain by the bargain?

Solution by the Rev. T. C. Simmons, M.A.

The conditions of the question are evidently the same as if A. were to receive in the event of a doublet the square of the numbers turned up plus their sum, and in the event of a triplet the cube of the numbers turned up plus their sum; but had *in any case* to forfeit the sum of the numbers turned up.

Now the number of ways in which a doublet of ones can occur is evidently 15, and the possible totals in this case for the three dice are 4, 5, 6, 7, 8, each total occurring 3 times. Hence A.'s expectation from these

is $$\tfrac{3}{216}\left[4^2+5^2+6^2+7^2+8^2+4+5+6+7+8\right]$$

or $$\tfrac{3}{216}\left[4\,.\,5+5\,.\,6+6\,.\,7+7\,.\,8+8\,.\,9\right].$$

So his expectation from a doublet of twos is

$$\tfrac{3}{216}\,[5\,.\,6+7\,.\,8+8\,.\,9+9\,.\,10+10\,.\,11],$$

and from doublets of threes, fours, fives, sixes,

$$\tfrac{3}{216}\,[7\,.\,8+8\,.\,9+10\,.\,11+11\,.\,12+12\,.\,13],$$

$$\tfrac{3}{216}\,[9\,.\,10+10\,.\,11+11\,.\,12+13\,.\,14+14\,.\,15],$$

$$\tfrac{3}{216}\,[11\,.\,12+12\,.\,13+13\,.\,14+14\,.\,15+16\,.\,17],$$

$$\tfrac{3}{216}\,[13\,.\,14+14\,.\,15+15\,.\,16+16\,.\,17+17\,.\,18] \text{ respectively.}$$

Hence his whole expectation from doublets is

$$\tfrac{3}{216}\,[4\,.\,5+6\,.\,7+15\,.\,16+17\,.\,18+2\,(5\,.\,6+9\,.\,10+12\,.\,13+16\,.\,17)$$
$$+3\,(7\,.\,8+8\,.\,9+10\,.\,11+11\,.\,12+13\,.\,14+14\,.\,15)]$$
$$=\tfrac{1}{72}\,[608+2\,.\,548+3\,.\,762]=\tfrac{665}{12} \text{ shillings.}$$

Again, the number of ways in which triplets can be obtained is 6, the respective totals being 3, 6, 9, 12, 15, 18. Hence A.'s expectation from these is $\tfrac{1}{216}\,[3^3+6^3+9^3+12^3+15^3+18^3+3+6+9+12+15+18]$

$$=\tfrac{1}{216}\,[27\,(1^3+2^3+3^3+4^3+5^3+6^3)+63]=\tfrac{665}{12} \text{ shillings,}$$

so that the whole value of his *positive* expectation is $\tfrac{665}{6}$ shillings or £5. 10*s*. 10*d*.

We have now to consider B.'s expectation. The number of ways in which any number n can be turned up is the coefficient of x^n in

$$(x+x^2+x^3+x^4+x^5+x^6)^3,$$

that is, in $x^3+3x^4+6x^5+10x^6+15x^7+21x^8+25x^9+27x^{10}$

$$+x^{18}+3x^{17}+6x^{16}+10x^{15}+15x^{14}+21x^{13}+25x^{12}+27x^{11},$$

so that the value of B.'s expectation is

$$\tfrac{21}{216}\,[1+3+6+10+15+21+25+27]=\tfrac{63}{6} \text{ shillings or } 10s.\ 6d.$$

Hence A. ought to have paid B. beforehand the difference between £5. 10*s*. 10*d*. and 10*s*. 6*d*., or £5. 0*s*. 4*d*., so that the bargain as it stands is slightly to the advantage of A.

7400. (By W. J. C. Sharp, M.A.)—Of the twenty-four lines which touch two of the circles inscribed and escribed to a triangle, eighteen are represented by the sides; construct the other six.

Solution by the Proposer.

Evidently each vertex is the external centre of similitude for the inscribed circle and the escribed circle opposite to that vertex, and the internal centre of similitude for the two adjacent escribed circles. The other

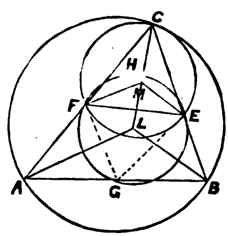
centres of similitude are the intersections of the internal bisectors of the angles with the opposite sides, which are the internal centres of similitude for the inscribed circle and each of the escribed circles, and the intersections of the external bisectors of the angles which are the external centres of similitude for the escribed circles adjacent to the angles bisected. Hence if, from each of the points where the bisectors, internal and external, of the angles meet the opposite sides, lines be drawn making angles with the bisectors equal to those which these make with side intersected, the lines so drawn are the six common tangents required.

Hence the three lines which touch the escribed and one of the escribed circles make angles with the side touched by the escribed circle equal to the difference of the angles at its extremities, and the three common tangents to two escribed circles make angles supplementary to those with same sides.

8114. (By Rev. T. C. Simmons, M.A.)—Prove that, according as a triangle is obtuse-angled, right-angled, or acute-angled, its nine-point circle will cut, touch, or lie entirely within, its circum-circle; also that, having given two circles, radii R and $\frac{1}{2}$R, not entirely external to each other, an infinite number of triangles may be described which shall have the one for circumscribed, and the other for nine-point, circle respectively.

Solutions by (1) Arthur Hill Curtis, LL.D., D.Sc.; (2) *the* Proposer.

1. Let ACB be the triangle, E, F, G the middle points of sides, the circle passing through which will be the nine-point circle. Describe the circle through E, F, and C: this circle will touch the circum-circle at C, as the tangent to each will be inclined to CF at an angle $= \angle$ CEF. Now, as FCE, FHE are the segments of equal circles and contain supplemental angles, it follows that, if $\angle$ C be acute, the latter segment lies altogether inside the former, and therefore *à fortiori* inside the circle ACB; the reverse is the case if $\angle$ C be obtuse, while, if $\angle$ C be a right angle, the segments coincide, and both touch the circum-circle at C. As the same results hold for the segments EG, GF, the theorem is proved.



If the circles ACB, EFG be given as in the figure, and *any* point C be assumed on the circumference of the larger circle, a triangle having C for one of its vertices can be constructed satisfying the conditions of the question. For, if L be the centre of the former circle, bisect CL in M; inflect to circle EFG, MF and ME, each $= \frac{1}{2}$R; draw CA, CB through F and E; then, bisecting AB in G, CEGF is a parallelogram, and therefore $\angle$ EGF $= \angle$ C, and, as EF is parallel to AB and $= \frac{1}{2}$AB, the circle on EF, of radius $\frac{1}{2}$R, that is the circle EFG, must contain angle $=$ C, and therefore pass through G; therefore ΔACB satisfies the required conditions.

2. *Otherwise* :—The first part of the theorem follows from the fact that, according as a triangle is acute-angled, obtuse-angled, or right-angled, its orthocentre, which is also the external centre of similitude of its nine-point and circumscribed circles, will lie within or without the latter, or on the perimeter.

For the second part, take O the external centre of similitude; through A on the perimeter of the larger circle draw AODd, and CDB perpen-

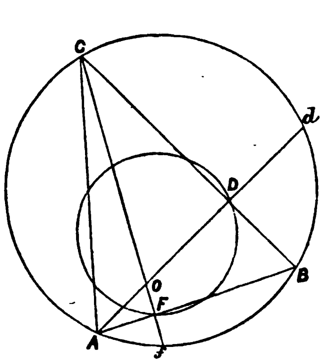


FIG. 2.

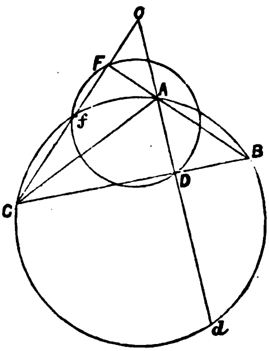


FIG. 3.

dicular at D, as in the figures, and let CO meet the circles at F and f. Then $AO \cdot OD = \frac{1}{2}AO \cdot Od = \frac{1}{2}CO \cdot Of = CO \cdot OF$, whence the triangles COD, AOF are similar, and AFO a right angle. Let AF now meet CD produced in B′; then the triangles CDO, ADB′ will be similar, so that $CD \cdot DB' = AD \cdot DO = AD \cdot Dd$; that is to say, B′ coincides with B, and the smaller circle passes through the foot of the perpendicular from C on BA. Similarly it passes through the foot of the perpendicular from B on CA, so that it is the nine-point circle of ABC. It will be seen, from this as well as from the preceding solution, that, when the circles are partly external, only a certain range of the perimeter of the larger circle is available for the vertex of an inscribed triangle of the kind required.

8102. (By Rev. T. R. TERRY, M.A.)—If the area bounded by the curve $x^3 - 3a\sqrt{2} \cdot xy + y^3 = 0$ and its asymptote revolve round the asymptote, prove that the volume generated is $\frac{2}{3}\pi a^3\sqrt{3}\,(9\sqrt{3} - 4\pi)$.

Solution by D. EDWARDES.

Turning the axes through 45°, the equation becomes

$$x(x^2 + 3y^2) - 3a(x^2 - y^2) = 0;$$

and transferring the origin to $(-a, 0)$, and putting $x = 4a\cos^2\phi$, we have $y = \frac{1}{2}a\sqrt{3}\sin 3\phi\sec\phi$; hence

$$V = \int_0^a 2y\,dx\,.\,2\pi x = \frac{64a^3\sqrt{3}\pi}{3}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\sin 3\phi\cos\phi\sin 2\phi\,d\phi$$

$$= \frac{16\pi a^3\sqrt{3}}{3}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}(1+\cos 2\phi-\cos 4\phi-\cos 6\phi)\,d\phi = \text{given result.}$$

7963. (By Professor Wolstenholme, M.A., Sc.D.)—A chord PQ of an ellipse is normal at P, and O is its pole; prove that, if a, b be the semi-axes, the maximum value of the angle QOP is $\tan^{-1}\left(\frac{a}{b}-\frac{b}{a}\right)$.

Solution by Rev. T. Galliers, M.A.; R. Knowles, B.A.; *and others.*

If the coordinates of P are $(a\cos\theta, b\sin\theta)$, the equation to PQ will be

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta} = a^2-b^2.$$

The coordinates of O, the pole of PQ, will be (h, k), where

$$h = a^3/(a^2-b^2)\cos\theta \text{ and } k = -b^3/(a^2-b^2)\sin\theta.$$

Now, in Wolstenholme's *Problems*, Ex. 991, it is proved that

$$\cos QOP = (OC^2-a^2-b^2)/SO\,.\,S'O,$$

or
$$\cot QOP = \tfrac{1}{2}(h^2+k^2-a^2-b^2)/(h^2b^2+k^2a^2-a^2b^2)^{\frac{1}{2}}$$

$$= \frac{a^4\sin^2\theta+b^4\cos^2\theta}{2ab(a^2-b^2)\sin\theta\cos\theta} = (a^4\tan\theta+b^4\cot\theta)/2ab(a^2-b^2).$$

For a maximum value of QOP, we have

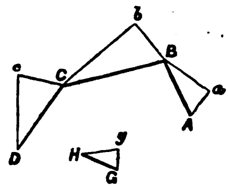
$$\frac{d}{d\theta}(\cot QOP) = 0;\ \text{or}\ a^4\sec^2\theta-b^4\operatorname{cosec}^2\theta = 0,\ \text{or}\ \tan\theta = b^2/a^2,$$

therefore
$$\cot QOP = 2a^2b^2/2ab(a^2-b^2),$$

or
$$\tan QOP = (a^2-b^2)/ab = \frac{a}{b}-\frac{b}{a}.$$

8131 & 7865. (By W. S. McCay, M.A.)—Given a plane polygon of n sides; show that the centre of gravity of any n points, similarly situated with respect to the sides in successive vectorial order, coincides with the centre of gravity of the vertices.

Solutions by (1) *the* Rev. T. C. SIMMONS, M.A.; (2) *the* PROPOSER; ARTHUR HILL CURTIS, LL.D., D.Sc.; *and others.*



1. Let AB, BC, CD be any three consecutive sides of the polygon, and a, b, c the corresponding points. Then the triangles AaB, BbC, CcD will be similar. Let triangles of the same kind be constructed all round the polygon, and then let equal particles, starting from A, B, C ... travel along the lines Aa, Bb, Cc ... with uniform velocities respectively proportional to AB, BC, CD ... The points a, b, c ... will all be reached simultaneously, and in the meantime the centre of gravity of all the particles will have travelled uniformly in a straight line from some position G to some other position g. Let the particles, each moving with the same velocity as before, now continue their course respectively along aB, bC, cD ...

The points B, C, D ... will all be reached at the same instant; and in the meantime, since the course of each particle has been diverted through the same angle, the course of the centre of gravity will have been likewise diverted through the same angle, and it will have travelled along a straight line gH with the same uniform velocity as before. Hence $g\text{H} : g\text{G} = a\text{B} : a\text{A}$, and $\angle \text{G}g\text{H} = \angle \text{A}a\text{B}$; or the triangle G$g$H is similar to A$a$B. But it is evident that the centre of gravity of all the particles is the same in the final as in the initial position; that is to say, H must coincide with G. Hence g likewise must coincide with G, or the centre of gravity of the n equal particles at a, b, c ... coincides with that of the same n equal particles at A, B, C ...

[This theorem, which includes Quest. 7865, solved independently on pp. 50, 51 of Vol. XLIII., can also be stated in the following more general form :—If similar polygons be similarly drawn on all the sides of any closed rectilineal figure, the centroid of all the points determined by their vertices will coincide with the centroid of the vertices of the original figure.]

2. *Otherwise* :—Let a, b, c ... be the vector sides in order, and $\rho_1, \rho_2, \rho_3 \dots$ the vectors to vertices from their centre of gravity (G), then

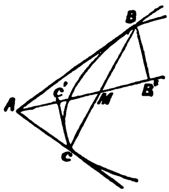
$$a+b+c+\dots = 0, \quad \rho_1+\rho_2+\rho_3+\dots = 0.$$

Let q be a quaternion operator whose axis is perpendicular to the plane of the polygon, then qa, qb, qc ... will be the vectors from the vertices to n points similarly situated to the vectors a, b, c ..., for they are proportional to the sides and equally inclined to them in order.

The vectors from G to these points are ρ_1+qa, ρ_2+qb, ρ_3+qc, ..., whose sum is $\rho_1+\rho_2+\rho_3+\dots+q\,(a+b+c+\dots)$, which is identically nothing, proving the theorem.

8107. (By SATIS CHANDRA RÂY.)—If θ, ϕ, ψ be the angles of inclination of any two tangents to a parabola, and of their chord of contact to the directrix, show that $\tan\theta+\tan\phi = 2\tan\psi$.

Solution by Professor NEUBERG; R. KNOWLES, B.A.; *and others.*



Soient AB, AC deux tangentes et BC leur corde de contact. La médiane AM du triangle ABC est parallèle à l'axe de la parabole. Menons BB', CC' perpendiculaires à AM. Nous aurons

$$2MC' = AC' - AB', \quad MC' = CC' \cot\psi,$$
$$AC' = CC' \cot\phi, \quad AB' = BB' \cot\theta = CC' \cot\theta, \text{ \&c.}$$

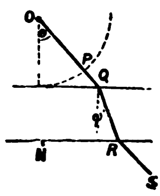
[Mr. KNOWLES remarks that the property is "true of any line at right angles to the axis," and Mr. SIMMONS states that the theorem is otherwise proved on p. 117 of WALTON's *Problems in Plane Coordinate Geometry*.]

7965. (By Professor STEGGALL, M.A.)—A spherical wave is refracted through a plane uniform plate of thickness t; show that (1) the equation of the refracted wave-surface is given by eliminating θ from

$$x = d\cos\theta - \frac{t\mu^2\cos\theta}{[\mu^2-\sin^2\theta]^{\frac12}}, \quad y = d\sin\theta - \frac{t(\mu^2-1)\sin\theta}{[\mu^2-\sin^2\theta]^{\frac12}},$$

where μ is the index of refraction; and hence (2) deduce a geometrical construction for any point in the wave-front.

Solution by A. GORDON; Professor NASH, M.A.; *and others.*



Let OPQRS be the course of any ray, v the velocity in air, v' in medium, $\dfrac{v}{v'} = \dfrac{\sin\theta}{\sin\phi'} = \mu$; we are given time in PQRS = constant = $\dfrac{\lambda}{v}$ suppose;

$$\therefore \quad (\text{RS}) + a\left(\frac{1}{\cos\theta} - 1\right) + \frac{\mu^2 t}{(\mu^2-\sin^2\theta)^{\frac12}} = \lambda.$$

Let x, y be coordinates of S, referred to N as origin. Then $x = t\tan\phi' + a\tan\theta + (\text{RS})\sin\theta$, $y = (\text{RS})\cos\theta$;

therefore $$x = (a+\lambda)\sin\theta + \frac{\sin\theta \,.\, t(1-\mu^2)}{(\mu^2-\sin^2\theta)^{\frac12}},$$

and $$y + \text{const.} = y' = (a+\lambda)\cos\theta - \frac{\mu^2 t\cos\theta}{(\mu^2-\sin^2\theta)^{\frac12}}.$$

8007. (By Professor ALEXANDER MACFARLANE, D.Sc., F.R.S.E.)—In a certain collection of objects having the known marks a, b, c, d, e, f, and the unknown marks x and y, those having the marks a and x, together with those

having the mark y but not the mark b, are identical with those having the mark c; and those having the mark d but not the mark x, excepting those without the mark e and without the mark y, are identical with those having the mark f. Determine those having the mark x, and those having the mark y.

Solution by the PROPOSER.

Let U denote the collection of objects, then the equations are

$$\mathrm{U}\,[ax+(1-b)\,y = c], \quad \mathrm{U}\,[d\,(1-x)-(1-e)(1-y) = f] \quad \ldots(1,\ 2).$$

Equation (2) may be put into the more convenient form

$$\mathrm{U}\,[dx-(1-e)\,y = d+e-f-1] \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Suppose U divided into the 64 classes formed by the presence or absence of the marks a, b, c, d, e, f. To find the expressions for x and for y in terms of a, b, c, d, e, f, suppose each of the classes in succession to be co-extensive with U; then, when a mark is present in the class, substitute 1, and, when absent, substitute 0, in the equations (1) and (3); the values of x and y so found are the coefficients of the class in the expressions for x and y.

Thus, for the term $abcdef$, we get $x = 1$ and $x = 0$, which is impossible, hence the class U$abcdef$ does not exist. For the term $abcdef'$, $x = 1$ and $x = 1$; hence x comprises the whole term $abcdef'$, and y comprises an indeterminate portion of it. For the term $abcde'f$, $x = 1$ and $x-y = -1$, which give y an integer value; but, as y is supposed not to be duplicate in any part, the class U$abcde'f$ cannot exist. For the term $abcde'f'$, $x = 1$ and $x-y = 0$; hence $x = 1$ and $y = 1$.

By continuing this method of solution, we find

$$\begin{aligned}\mathrm{U}x = {} & \text{the whole of U}\,[abcdef' + abcde'f' + abcd'ef' + abcd'e'f' \\ & \qquad + ab'cdef' + a'bc'def' + a'b'cdef' + a'b'cde'f' + a'b'c'def'] \\ & + \text{a portion of U}\,[a'bc'd'ef' + a'bc'd'e'f' + a'b'cd'ef' + a'b'cd'e'f' + a'b'c'd'ef'] \\ & + \text{a portion of U}ab'cd'ef' + \text{a portion of U}a'bc'de'f'.\end{aligned}$$

$$\begin{aligned}\text{And }\mathrm{U}y = {} & \text{the whole of U}\,[abcde'f' + abcd'e'f' + abc'de'f + abc'd'e'f' + ab'cdef \\ & \qquad + ab'cde'f + ab'cd'e'f' + a'bc'de'f + a'bc'd'e'f' + a'b'cdef + a'b'cdef' \\ & \qquad + a'b'cde'f + a'b'cde'f' + a'b'cd'ef' + a'b'cd'e'f'] \\ & + \text{a portion of U}\,[abcdef' + abcd'ef' + abc'def + abc'd'ef' \\ & \qquad + a'bc'def + a'bc'def' + a'bc'd'ef'] \\ & + \text{the complementary portion of U}ab'cd'ef' \\ & + \text{the same portion of U}a'bc'de'f'.\end{aligned}$$

The expression for Ux may be simplified to

$$\begin{aligned}\mathrm{U}x = {} & \text{the whole of U}\,\{abc + [(b'c + a'c')\,e + a'b'ce']\,d\}\,f' \\ & + \text{a portion of U}\,[bc' + b'\,(c + c'e)]\,a'd'f' \\ & + \text{the complementary portion of U}ab'cd'ef' \\ & + \text{the same portion of U}a'bc'de'f'.\end{aligned}$$

To express this in unambiguous English, it is necessary to punctuate with mathematical parentheses, thus—The objects having the mark x are identical with the objects which are without f, and {either with a, b and e or with d and [either without a, b and e, but with c or with e, and (either without a and c or without b and with c)]}; together with a portion of those which are without a, d, and f and [either with b, but without c or without b, and (either with c or without c and with e)]; together with the complementary and identical portions. The expression for y can also be simplified, but not to the same extent.

In *Laws of Thought*, Boole attempts to prove that the only possible coefficients are *all*, *none*, *apart*, and *impossible*; but the above solution shows that there are other possible coefficients, as *complementary part* and *identical part*, and the investigation also discriminates between the different kinds of impossibility. Boole's method is to solve and then to substitute; mine is to substitute and then to solve.

7942. (By W. J. C. Sharp, M.A.)—If P be the centroid of three weights a, b, c, at the points A, B, C, prove that

$$\frac{a\,.\,\mathrm{AP}}{\sin \mathrm{BPC}} = \frac{b\,.\,\mathrm{BP}}{\sin \mathrm{CPA}} = \frac{c\,.\,\mathrm{CP}}{\sin \mathrm{APB}}.$$

Solution by Rev. J. L. Kitchin, M.A.; J. O'Regan; *and others.*

If triangles BPC, APC, APB be proportional to a, b, c, then P will be the centroid of these triangles. Let them be la, lb, lc. Now

$$la = \tfrac{1}{2}\mathrm{AP}\,.\,\mathrm{PC}\sin \mathrm{BPC};$$

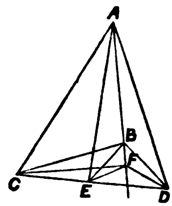
therefore $$\frac{la\,\mathrm{AP}}{\sin \mathrm{BPC}} = \frac{\mathrm{AP}\,.\,\mathrm{BP}\,.\,\mathrm{CP}}{2} = \frac{lb\,.\,\mathrm{BP}}{\sin \mathrm{APC}} = \frac{lc\,.\,\mathrm{PC}}{\sin \mathrm{APB}}$$ by symmetry,

therefore $$\frac{a\,.\,\mathrm{AP}}{\sin \mathrm{BPC}} = \frac{b\,.\,\mathrm{BP}}{\sin \mathrm{APC}} = \frac{c\,.\,\mathrm{PC}}{\sin \mathrm{APB}}.$$

[This is a solution of a particular case of the question, and is then really tantamount to the known case of the centroid of the perimeter of a triangle similar to ABC, the mid-points of whose sides are at A, B, C respectively. The property is more general, and a, b, c are not intended to be proportional to the sides of ABC, but to denote any weights. The property follows at once from taking moments about CP, &c.]

8051. (By Professor Genese, M.A.) — ABCD is any tetrahedron; prove that the plane bisecting the dihedral angle between the planes ABC and ABD divides CD in the ratio of the distances of C and D from AB.

Solution by EMILY PERRIN; D. BIDDLE; *and others.*



Let the plane bisecting the angle between ABC and ABD meet CD in E. Draw EF perpendicular to AB, and join FC, FD. Suppose a plane through EF perpendicular to AB, and another plane through CD parallel to AB, and let these together meet the planes AFC, AFD in points (say) Q, R. Then FE bisects the angle QFR, and EQ : ER = QF : RF (Euc. VI. 3). The foregoing planes will also form with the plane CFD, similar triangles, ECQ, EDR, in which CE : ED = EQ : ER. Therefore CE : ED = QF : RF. But QF, RF are evidently (Euc. XI. 16) the distances from AB of C and D respectively. Hence, &c.

7934. (By ÂSÛTOSH MUKHOPÂDHYÂY, B.A., F.R.A.S.)—The centre of pressure of a triangular lamina immersed in a homogeneous fluid coincides with the centre of the nine-point circle; prove that the depths to which the mid-points of the sides are immersed below the free surface are proportional to the sides of the pedal triangle.

Solution by DR. CURTIS; REV. T. GALLIERS, M.A.; *and others.*

In the *Messenger of Mathematics* (New Series, No. 12, 1872), it is shown that in this case $H_1 : H_2 : H_3 = \sin 2A : \sin 2B : \sin 2C$; but the sides of the pedal triangle are also proportional to $\sin 2A$, $\sin 2B$, $\sin 2C$, for, if α denote the side of the pedal triangle opposite angle A, then α is a chord of the circle having the side a as diameter, and, as in this circle it subtends an angle $= \frac{1}{2}\pi - A$, $\alpha = a \cos A$, but, if R be the radius of the circumscribing circle of the given triangle, $a = 2R \sin A$, $\therefore \alpha = 2R \sin A \cos A = R \sin 2A$, therefore $\alpha : \beta : \gamma = \sin 2A : \sin 2B : \sin 2C = H_1 : H_2 : H_3$.

7987. (By the Rev. T. R. TERRY, M.A.)—Four spheres whose radii are a, b, c, d, respectively, are such that each touches the other three externally. In the space between these four, another sphere, radius r, is described touching all four externally. Show that

$$\frac{1}{r^2} - \frac{1}{r}\Sigma\left(\frac{1}{a}\right) + \Sigma\left(\frac{1}{a^2}\right) - \Sigma\left(\frac{1}{ab}\right) = 0.$$

Solution by R. LACHLAN, B.A.

Let A, B, C, D be the centres of the four given spheres, and O the centre of the sphere which touches them; then we have the relation

$$\begin{vmatrix} -1, & \cos AOB, & \cos AOC, & \cos AOD \\ \cos BOA, & -1, & \cos BOC, & \cos BOD \\ \cos COA, & \cos COB, & -1, & \cos COD \\ \cos DOA, & \cos DOB, & \cos DOC, & -1 \end{vmatrix} = 0,$$

and $$\cos AOB = \frac{AO^2 + OB^2 - AB^2}{2OA \,.\, OB} = \frac{r^2 + ra + rb - ab}{OA \,.\, OB};$$

also $$\begin{vmatrix} -OA^2, & OA \,.\, OB \cos AOB, \ldots \\ OA \,.\, OB \cos BOA, & -OB^2, \ldots \end{vmatrix} = 0;$$

therefore $$\begin{vmatrix} -r^2 - 2ra - a^2, & r^2 + ar + br - ab, \ldots \\ r^2 + ar + br - ab, & -r^2 - 2br - b^2, \ldots \\ \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \end{vmatrix} = 0,$$

which reduces to $$\begin{vmatrix} 0, & \frac{1}{r}, & \frac{1}{a}, & \frac{1}{b}, & \frac{1}{c}, & \frac{1}{d} \\ \frac{1}{r}, & -1, & 1, & 1, & 1, & 1 \\ \frac{1}{a}, & 1, & -1, & 1, & 1, & 1 \\ \frac{1}{b}, & 1, & 1, & -1, & 1, & 1 \\ \frac{1}{c}, & 1, & 1, & 1, & -1, & 1 \\ \frac{1}{d}, & 1, & 1, & 1, & 1, & -1 \end{vmatrix} = 0,$$

or $$16\left\{\frac{1}{r^2} + \Sigma\left(\frac{1}{a^2}\right)\right\} + 2\left\{\frac{1}{r}\Sigma\left(\frac{1}{a}\right) + \Sigma\left(\frac{1}{ab}\right)\right\}(-16+8) = 0;$$

whence the required result immediately follows.

8151. (By E. RUTTER.)—From any point P in the bisector of the angle A in a triangle ABC, perpendiculars PA′, PB′, PC′ are drawn to the three sides; prove that PA′ and B′C′ intersect on the median from A.

Solution by G. HEPPEL, M.A.; R. KNOWLES, B.A.; *and others.*

Take A as origin, AP as axis of x. Let $AP = h$, and let AD be the median. Then P is $[h, 0]$, B is $[a, ma]$, C is $[b, -mb]$,

B′C′ is $u \equiv (1+m^2)\,x - h = 0$, PA′ is $v \equiv m\,(a+b)\,y + (a-b)\,x - (a-b)\,h = 0$;

AD is $w \equiv (a+b)\,y - m\,(a-b)\,x = 0$, and $u\,(a-b) - v + mw \equiv 0$;

hence the three lines meet in a point.

8116. (By Professor NEUBERG. Generalization of Quest. 8047.) —Le lieu d'un point M tel que les droites AM, BM, CM, DM rencontrent les faces du tétraèdre ABCD aux sommets d'un tétraèdre A′B′C′D′ triple de ABCD, se compose du plan à l'infini et d'une surface du troisième ordre. Trouver le théorème analogue dans le triangle.

Solution by the PROPOSER.

Soient $(\alpha, \beta, \gamma, \delta)$ les coordonnées barycentriques d'un point quelconque M, et A′, B′, C′, D′ les points de rencontre des droites AM, BM, CM, DM avec les faces correspondantes du tétraèdre. Si nous supposons $\alpha+\beta+\gamma+\delta = 1$, les coordonnées de A′, B′, C′, D′ seront

$$\left(0,\ \frac{\beta}{1-\alpha},\ \frac{\gamma}{1-\alpha},\ \frac{\delta}{1-\alpha}\right),\ \left(\frac{\alpha}{1-\beta},\ 0,\ \frac{\gamma}{1-\beta},\ \frac{\delta}{1-\beta}\right),$$
$$\left(\frac{\alpha}{1-\gamma},\ \frac{\beta}{1-\gamma},\ 0,\ \frac{\delta}{1-\gamma}\right),\ \left(\frac{\alpha}{1-\delta},\ \frac{\beta}{1-\delta},\ \frac{\gamma}{1-\delta},\ 0\right).$$

Le déterminant formé avec ces coordonnées représente le rapport des volumes A′B′C′D′, ABCD. En calculant le déterminant, on trouve

$$\frac{\text{A}'\text{B}'\text{C}'\text{D}'}{\text{ABCD}} = \frac{3\alpha\beta\gamma\delta}{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Si le volume A′B′C′D′ doit rester constant, le point M décrit une surface du quatrième ordre. Supposons A′B′C′D′ = 3ABCD; la formule (1) devient $(1-\alpha)(1-\beta)(1-\gamma)(1-\delta) = \alpha\beta\gamma\delta$, ou sous forme homogène

$$\Sigma\alpha\,(\Sigma\alpha\,.\,\Sigma\alpha\beta - \Sigma\alpha\beta\gamma) = 0.$$

Donc le lieu de M est alors le plan à l'infini (Question 8047) et une surface du troisième ordre.

Dans le cas du triangle, le lieu d'un point M tel que les droites AM, BM, CM rencontrent BC, CA, AB aux sommets d'un triangle A′B′C′ double de ABC, est la droite à l'infini et l'ellipse circonscrite à ABC qui a pour centre le centre de gravité. On prend pour équation de ce lieu $(1-\alpha)(1-\beta)(1-\gamma) = \alpha\beta\gamma$, ce qui suppose les triangles ABC, A′B′C′ de sens contraires.

8155. (By Professor NEUBERG.)—Lorsque la base BC d'un triangle ABC et l'angle de BROCARD sont donnés, prouver que le sommet A décrit deux circonférences.

Solution by SAMUEL ROBERTS, M.A.

Let A be a Brocard-point of the triangle A′BC, and

$$\angle \text{ACB} = \text{ABA}' = \text{AA}'\text{C} = w.$$

Then if the sides of the triangle ABC are denoted, as usual, by a, b, c, those of A′BC by α, β, γ, and AA′ = k, we have

$$b^2+a^2-2ab\cos w = c^2,\quad c^2+\gamma^2-2c\gamma\cos w = k^2,\quad k^2+\beta^2-2k\beta\cos w = b^2,$$

and, consequently, $a^2+\beta^2+\gamma^2-2(ab+c\gamma+k\beta)\cos w = 0$.

But $(ab+c\gamma+k\beta)\sin w$ = twice the area of the triangle A′BC; so that

$$a^2+\beta^2+\gamma^2-\cot w\left[2(a^2\beta^2+a^2\gamma^2+\beta^2\gamma^2)-a^4-\beta^4-\gamma^4\right]^{\frac{1}{2}}=0,$$

or $(\beta^2+\gamma^2)^2\sin^2 w+(\beta^2-\gamma^2)^2\cos^2 w+a^4+2a^2(\beta^2+\gamma^2)(\sin^2 w-\cos^2 w)=0$.

If we write for β^2, x^2+y^2, and for γ^2, $(x-a)^2+y^2$, this becomes

$$\sin^2 w\,(x^2+y^2-ax+a^2)^2-a^2y^2\cos^2 w=0,$$

representing two circles which can be readily constructed.

NOTE ON TRIANGLE-NOTATION. *By the* EDITOR.

In order to harmonize, so far as possible, the various sets of letters used to designate the prominent points relating to a triangle, and with a view to meet suggestions often made to adopt some uniform notation for these points, we have consulted several correspondents who take an interest in the question, and submit hereunder two out of many proposals that have been sent in answer to these inquiries.

1. Professor HUDSON—whose views many accept without alteration—considers the following letters to be the best for the purpose:—

	T orthocentre,
A, B, C, vertices,	G mass-centre,
D, E, F, mid-points of sides,	S circumcentre,
P, Q, R, feet of perpendiculars,	H mid- (*i.e.* nine-point) centre,

O, O_1, O_2, O_3, in- and ex-centres,

L, M, N, points of contact of in-circle,

$\left.\begin{matrix} L_1,\ M_1,\ N_1 \\ L_2,\ M_2,\ N_2 \\ L_3,\ M_3,\ N_3 \end{matrix}\right\}$ points of contact of ex-circles,

p, *q*, *r*, where AP, BQ, CR meet circumcircles,
U, V, W, mid-points of AT, BT, CT,
X, Y, Z, where any transversal cuts the sides.

If a name is wanted for the line on which the orthocentre, the mid-centre, the mass-centre, and the circumcentre of a triangle lie, I propose that the name should be applied to the line limited by the orthocentre and the circumcentre. The length of this line vanishes when the triangle is equilateral, and is short for a triangle that is nearly equilateral. It lies wholly within or extends beyond the triangle at both ends, according as the triangle is acute or obtuse-angled; in the separating case of a right-angled triangle, its extremities lying on the triangle. Since its length is a rude measure of the inequality of the sides of a triangle, and the position of its extremities indicates the character of the triangle as determined by its angles, I suggest that it be called the *scalene line* of the triangle. If this be considered a good name, I should then go on to suggest that the ratio of the scalene line to the semi-sum of the sides shall be called the

scalenity of the triangle; this ratio expressed in terms of the angles is

$$\sqrt{(1-8\cos A\cos B\cos C)} : \sin A+\sin B+\sin C.$$

2. Mr. SIMMONS writes as follows :—

There are so many advantages in retaining the initial letter wherever possible, that O seems preferable for orthocentre, I for in-centre, E_1, E_2, E_3 for ex-centres, and G for centroid, while S seems to be very commonly used for circumcentre (compare the Syllabus of the Association for the Improvement of Geometrical Teaching). In some respects, K might be better for circumcentre and S for Symmedian-point; but usage appears here to be getting too firmly fixed to hope for any alteration. As to nine-point-centre, it is not easy to see what is gained by calling it mid-centre; the latter name would seem at least equally appropriate for Symmedian-point. If the old title be retained, N would of course be the most natural letter.

There are, I think, two objections to Professor HUDSON's title, *scalene-line*. In the first place, it would be equally applicable either to the line joining circumcentre and in-centre, or to that joining centroid and in-centre, and so is ambiguous. Secondly, its length is not always "a rude measure of the inequality of the sides." For, take two isosceles triangles, the first having two sides each double of the third, and the other, the limit-triangle, having two sides each half of the third. Most people would probably estimate the *scalenity* of both triangles as about equal; whereas the so-called *scalene-line* is very small in the first case, and infinite in the second. A better title, both unambiguous and useful as a *memoria technica*, might, I think, be CONG-line (since it passes through Circumcentre, Orthocentre, Nine-point-centre, and Gravity-centre). It may be noted that triangles having the same Symmedian-point-axis have the same *Cong*-line, the equation of the latter referred to any one of the triangles being

$$\alpha(b^2-c^2)\cos A+\beta(c^2-a^2)\cos B+\gamma(a^2-b^2)\cos C = 0.$$

8075. (By Professor COCHEZ.)—Démontrer que dans un triangle quelconque l'on a

$$\left(\frac{\cos A}{c\sin B}+\frac{\cos B}{a\sin C}+\frac{\cos C}{b\sin A}\right)\left(\frac{1}{c\sin B}+\frac{1}{a\sin C}+\frac{1}{b\sin A}\right)$$
$$=2\left(\frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab}\right).$$

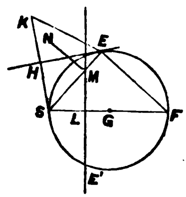
Solution by CH. COSTA; G. G. STORR, B.A.; *and others.*

Il a été démontré (Quest. 7699) que le 1^er facteur se réduisait à $\frac{1}{R}$. Quant au 2^ème facteur il se réduit aisément à $\frac{1}{h}+\frac{1}{h'}+\frac{1}{h''}$. Or cette somme est égale à r^{-1}, r étant le rayon du cercle inscrit. Donc le 1^er membre est égal à $1/Rr$.

Mais $\quad \frac{1}{Rr}=\frac{4p}{abc}=\frac{2(a+b+c)}{abc}$, ou bien $\frac{1}{Rr}=2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)$.

8133. (By Emily Perrin.)—Through the pole S of a cardioid there are drawn three circles touching the cardioid, and intersecting again in A, B, C; prove that (1) their centres will be concyclic with S; (2) A, B, C will be collinear; (3) the circles on SA, SB, SC as diameters will have a common chord.

Solution by A. H. Curtis, LL.D., D.Sc.; *the* Proposer; *and others.*



Let SEFE′ be the circle, of radius r, from which is generated the cardioid whose cusp is at S, SF being a diameter and G the centre; let EH be any tangent to this circle, SK the perpendicular from S on EH, and HK = SH, then K is a point on the cardioid and KE the corresponding normal, and therefore the circle with centre E and radius EK ≡ ES will touch the cardioid at K, and pass through S; hence (1) the locus of the centres of all circles described as in the question is the circle SEFE′.

(2) Let SE be intersected in M by LM perpendicular to SF at L, any assumed point on SF, which, for simplicity, may be supposed the middle point of SG, then as $\angle \text{SEF} = \frac{1}{2}\pi = \angle \text{MLF}$, $\text{SE} . \text{SM} = \text{SF} . \text{SL} = r^2$, therefore the line MN, perpendicular to SE, is the inverse of the circle of radius ES described round E, therefore MN and two similarly constructed lines will form a triangle whose vertices A′, B′, C′ are the three points inverse to A, B, C, the intersections of the three circles of the question; but, as the feet of the three perpendiculars let fall from S on these three lines lie on the straight line LM, S is a point on the circumcircle of the triangle A′B′C′, therefore the three points A, B, C, which lie on the inverse of this circle, are collinear.

(3) Again, the perpendiculars at A′, B′, C′ to the three lines SA′, SB′, SC′ must meet in N′, the point on the circumcircle of triangle A′B′C′ diametrically opposite to S, therefore their three inverses, the circles on SA, SB, SC as diameters, meet in N, the point inverse to N′; therefore SN is a chord common to them.

8055 & 8101. (By Rev. T. C. Simmons, M.A.)—(8055.) Three tangents being drawn at random to a given circle, show that the odds are 3 to 1 against the circle being *in*scribed in the triangle formed by them.

(8101.) If a triangle be formed by joining three points taken at random in the circumference of a given circle, prove, by elementary geometry or otherwise, that the odds are 3 to 1 against its being acute-angled.

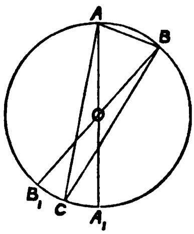
Solutions by (1) A. Gordon, G. Heppel, M.A., *and others*; (2) D. Biddle; (3) *the* Proposer.

1. Let the points of contact of the tangents in (8055), or the vertices of the triangle in (8101), be denoted by A, B, C (Fig. 1), and let A′, B′ be the

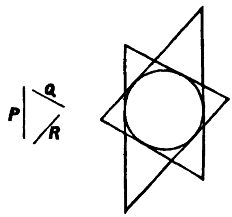
opposite extremities of diameters through A, B. Then A may be supposed fixed; and, putting $\angle AOB = \theta$, the chance of B lying between θ and $\theta + d\theta$ is $\frac{d\theta}{\pi}$, in which case, either for an inscribed circle in (8055) or for an acute-angled triangle in (8101), we must take C between A′ and B′, the chance of which is $\theta / 2\pi$. Hence the whole chance is

$$\int_0^\pi \frac{d\theta}{\pi} \cdot \frac{\theta}{2\pi} = \frac{1}{4},\text{ or the odds are 3 to 1 against.}$$

2. *Otherwise* :—(Fig. 1.) A, the first point, may be anywhere. B, also, may be anywhere, the arc AB never exceeding a semi-circle. The diameters AA_1, BB_1 being imagined drawn, A_1B_1 represents the arc on which C must be found, either for an inscribed circle in (8055) or for an acute-angled triangle in (8101). But $A_1B_1 = AB$, and the average length of $AB = \frac{1}{4}$ of the circumference. Hence $P = \frac{1}{4}$, and the odds $= 3 : 1$ against.



(Fig. 1.)



(Fig. 2.)

3. *Otherwise* :—(Fig. 2.) The following proof of (8055) may be of interest on account of its extreme simplicity.

Draw in the plane of the paper any three random lines P, Q, R; and the six tangents to the circle parallel thereto. Then, of the eight triangles formed by the latter, it is evident that the circle will be escribed to six and inscribed in two; and, as this is true whatever be the original directions of P, Q, R, the required result follows. A similar method of proof applicable to (8101) could easily be given.

7910. (By Rev. T. R. Terry, M.A.)—Two equal rods OA and OB of length $2a$ and mass m are rigidly joined at O; OA is horizontal and OB is vertical and downwards. The ends of a third rod PQ of length $2a$ and mass M can slide freely along OA and OB respectively, and the whole system is rotating round OB as a fixed vertical axis. If PQ is making small oscillations about a position of equilibrium in which it makes an angle $\frac{1}{4}\pi$ with the vertical, show that the length of the simple equivalent pendulum is $\frac{8a}{9} \cdot \frac{4m + 3M}{4m + 7M}$.

Solution by D. EDWARDES; Professor N. SARKAR, M.A.; *and others.*

In the general motion of the system, let θ be the inclination to the vertical of the rod PQ at the time t, and ϕ the angle between the plane of the rods and some fixed vertical plane. The components of the velocity of the centroid of PQ will be $a\sin\theta\,\dfrac{d\phi}{dt}$, $a\dfrac{d\theta}{dt}$, and the kinetic energy of the system will be

$$\tfrac{1}{2}\mathrm{M}\left[a^2\sin^2\theta\dot{\phi}^2 + a^2\dot{\theta}^2 + \tfrac{1}{3}a^2\sin^2\theta\dot{\phi}^2 + \tfrac{1}{3}a^2\dot{\theta}^2\right] + \tfrac{2}{3}ma^2\dot{\phi}^2.$$

The equation of energy will therefore be

$$\mathrm{M}\dot{\theta}^2 + (\mathrm{M}\sin^2\theta + m)\,\dot{\phi}^2 = \tfrac{3}{2}\frac{\mathrm{M}g}{a}\cos\theta + \mathrm{C}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Suppose that initially, $\theta = \alpha$, $\dfrac{d\theta}{dt} = 0$, $\dfrac{d\phi}{dt} = w$. Then, since the whole momentum about the vertical axis remains unchanged, we have

$$(\mathrm{M}\sin^2\theta + m)\,\dot{\phi} = (\mathrm{M}\sin^2\alpha + m)\,w,$$

and equation (1) becomes

$$\mathrm{M}\dot{\theta}^2 + \frac{(\mathrm{M}\sin^2\alpha + m)^2w^2}{\mathrm{M}\sin^2\theta + m} = \tfrac{3}{2}\frac{\mathrm{M}g}{a}\cos\theta + (\mathrm{M}\sin^2\alpha + m)\,w^2 - \tfrac{3}{2}\frac{\mathrm{M}g}{a}\cos\alpha,$$

$$\text{or }\ \mathrm{M}\dot{\theta}^2 = \tfrac{3}{2}\frac{\mathrm{M}g}{a}\cos\theta - \frac{(\mathrm{M}\sin^2\alpha + m)^2w^2}{\mathrm{M}\sin^2\theta + m} + (\mathrm{M}\sin^2\alpha + m)\,w^2 - \tfrac{3}{2}\frac{\mathrm{M}g}{a}\cos\alpha.$$

Hence, for a position of stable equilibrium,

$$\frac{2\,(\mathrm{M}\sin^2\alpha + m)^2w^2\cos\theta}{(\mathrm{M}\sin^2\theta + m)^2} = \tfrac{3}{2}\frac{\mathrm{M}g}{a},$$

and, if $\theta = \frac{1}{3}\pi$, $\quad(\mathrm{M}\sin^2\alpha + m)^2w^2 = \tfrac{3}{2}\dfrac{\mathrm{M}g}{a}(\tfrac{3}{4}\mathrm{M} - m)^2.$

The equation of motion is now

$$\dot{\theta}^2 + \frac{3g}{2a}\,\frac{(\frac{3}{4}\mathrm{M} + m)^2}{\mathrm{M}\sin^2\theta + m} = \frac{3g}{2a}\cos\theta + \text{constant}.$$

Differentiating, and then putting

$$\theta = \tfrac{1}{3}\pi + \beta,\quad \sin\theta = \tfrac{1}{2}(\sqrt{3} + \beta),\quad \cos\theta = \tfrac{1}{2}(1 - \sqrt{3}\beta),$$

and, neglecting higher powers of β, we have

$$\ddot{\beta} + \frac{9g}{8a}\cdot\frac{4m + 7\mathrm{M}}{4m + 3\mathrm{M}}\,\beta = 0,\ \text{and therefore, \&c.}$$

7347. (By W. J. C. SHARP, M.A.)—If the chords of contact of the tangents whose intersection determines a focus, be called directrices, every line whose satellite conic with respect to a bicircular quartic is a circle, is a directrix.

Solution by the PROPOSER.

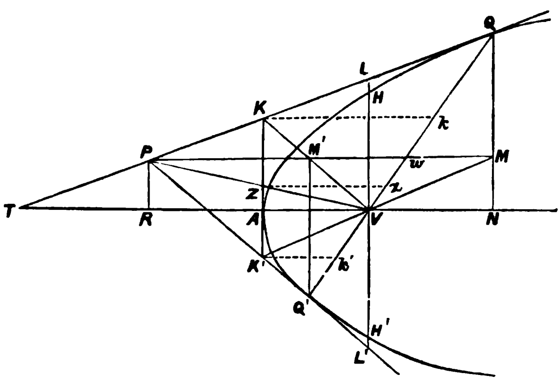
If a, b, c, d be the four tangents to the bicircular quartic at the points where it meets the line l, and G be the satellite conic, *i.e.* the conic on which the other eight points in which the tangents meet the curve must lie; the equation to the bicircular quartic may be put in the form $abcd = l^2\,G$.

Then, if G be a circle and the axes be rectangular, the biquadratic portion of $abcd$ (the equation to the parallels to the tangents through the origin) must be divisible by x^2+y^2, and therefore two of the tangents must be drawn from the circular points, and determine a focus, the directrix corresponding to which is l.

7500. (By the Rev. T. C. SIMMONS, M.A.)—Perpendiculars QM, Q′M′ are drawn from the extremities of a chord QQ′ of a parabola on the diameter through its pole P. If QQ′ meet the axis in V, prove that VM, VM′ form with PQ, PQ′ a parallelogram whose other angular points lie on the tangent at the vertex; also that the two portions of the ordinate through V situated between PQ, PQ′ and the curve are equal.

Solution by the PROPOSER.

Let KK′ be the tangent at the vertex. Produce QP to meet axis in T, join VK, VK′, VM, VM′, and VP meeting KK′ in Z; draw PR, QMN perpendiculars on axis, and Kk, $K'k'$, Zz, Pw parallel to axis.



Then, since P is the pole of QQ′, PR is the polar of V, whence R is the pole of HVH′, therefore RA = AV, PZ = ZV, and $wz = z$V. Now Qw = wQ′, therefore VQ − Vw = VQ′ + Vw, that is, 2Vw = VQ − VQ′ or 4Vz = 2Vk − 2Vk', therefore 2Vz = Vk − Vk', that is, z is the mid-

point of kk', or Z the mid-point of KK′, whence, noting that PZ = ZV, it follows that PKVK′ is a parallelogram.

Also since KK′ bisects both TN and RV, we have TR = VN, showing that the triangles TRP and VNM are equal in all respects; therefore VM is parallel to TPQ, *i.e.*, M lies on K′V. So M′ lies on KV.

Again, since KZ = ZK′, LV = VL′; but HV = VH′, therefore the intercepted portions LH, L′H′ are equal.

3043 & 3981. (By Professor Wolstenholme, M.A., Sc.D.) – A bag contains m white balls and n black balls, and balls are to be drawn from it so long as all drawn are of the same colour; if these be white, A pays B x shillings for the first, rx for the second, $\frac{1}{2}[r(r+1)x]$ for the third, $\frac{1}{6}[r(r+1)(r+2)x]$ for the fourth, and so on; but, if they be black, B pays A y shillings for the first, ry for the second, and so on. Prove that the value of A's expectation at the commencement of the drawing is

$$\frac{(m+n+r-1)!\,m!\,n!}{(m+n)!\,(m+r)!\,(n+r)!}\left[n(n+1)\ldots(n+r)\,y-m(m+1)\ldots(m+r)\,x\right].$$

Solution by the Rev. T. C. Simmons, M.A.

The chance that the first ball is black $=\dfrac{n}{m+n}$, the chance that the second likewise is black $=\dfrac{n(n-1)}{(m+n)(m+n-1)}$, and so on. Hence the value of A's expectation with respect to favourable cases is y shillings multiplied by

$$\frac{n}{m+n}+\frac{n(n-1)}{(m+n)(m+n-1)}\cdot r+\frac{n(n-1)(n-2)}{(m+n)(m+n-1)(m+n-2)}\cdot\frac{r(r+1)}{1\,.\,2}+\ldots$$
$$\ldots+\frac{m!\,n!}{(m+n)!}\cdot\frac{(r+n-2)!}{(n-1)!\,(r-1)!}.$$

Multiplying by $(m+n)!$ and dividing by $m!$ times $n!$, this becomes

$$\frac{(m+1)(m+2)\ldots(m+n-1)}{1\,.\,2\,.\,3\ldots(n-1)}+\frac{(m+1)(m+2)\ldots(m+n-2)}{1\,.\,2\,.\,3\ldots(n-2)}\cdot r$$
$$+\frac{(m+1)(m+2)\ldots(m+n-3)}{1\,.\,2\,.\,3\ldots(n-3)}\cdot\frac{r(r+1)}{1\,.\,2}+\ldots+\frac{r(r+1)\ldots(r+n-2)}{1\,.\,2\,.\,3\ldots(n-1)},$$

which is equal to the coefficient of x^{n-1} in $(1-x)^{-(m+1)}\times(1-x)^{-r}$, that is, to the same coefficient in $(1-x)^{-(m+r+1)}$, which is equal to $\dfrac{(m+n+r-1)!}{(m+r)!\,(n-1)!}$.

Hence the value in shillings of A's *positive* expectation is

$$\frac{(m+n+r-1)!\,m!\,n!}{(m+n)!\,(m+r)!\,(n-1)!}\cdot y=\frac{(m+n+r-1)!\,m!\,n!}{(m+n)!\,(m+r)!\,(n+r)!}\cdot n(n+1)\ldots(n+r)\,y.$$

Interchanging m and n and substituting x for y, we obtain the value of his *negative* expectation

$$= -\frac{(m+n+r-1)!\,m!\,n!}{(m+n)!\,(m+r)!\,(n+r)!}\cdot m(m+1)\dots(m+r)\,x,$$

whence, by addition, the result stated in the question immediately follows.

[For the Proposer's solution, see Vol. xx., p. 69.]

7989. (By W. J. Greenstreet, B.A.)—Show that the locus of the orthocentre of the triangle of which two semi-conjugate diameters of an ellipse are adjacent sides is

$$2(a^2x^2+b^2y^2)^3 = (a^2-b^2)^2(b^2y^2-a^2x^2)^2.$$

Solution by Rev. T. Galliers, M.A.; R. Knowles, B.A.; *and others.*

Let P, Q be the ends of conjugate diameters, their coordinates being $(a\cos\theta,\ b\sin\theta)$ and $(-a\sin\theta,\ b\cos\theta)$ respectively; the orthocentre required is the point of intersection of the normals at P and Q; hence, eliminating θ from the equations

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta} = a^2-b^2 = -\frac{ax}{\sin\theta}-\frac{by}{\cos\theta},\quad \frac{ax+by}{\cos\theta} = \frac{by-ax}{\sin\theta} = (2a^2x^2+2b^2y^2)^{\frac{1}{2}},$$

we have $\dfrac{ax}{ax+by}-\dfrac{by}{by-ax} = \dfrac{a^2-b^2}{(2a^2x^2+2b^2y^2)^{\frac{1}{2}}}$, which gives the result

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8011. (By Professor Cochez.)—Démontrer la somme

$$\tan\frac{x}{2}\sec x+\tan\frac{x}{2^2}\sec\frac{x}{2}+\dots+\tan\frac{x}{2^{n+1}}\sec\frac{x}{2^n} = \tan x-\tan\frac{x}{2^{n+1}}.$$

Solution by G. G. Storr, B.A.; H. S. Jones; *and others.*

Last term $=\sin\left(\dfrac{x}{2^n}-\dfrac{x}{2^{n+1}}\right)\sec\dfrac{x}{2^{n+1}}\sec\dfrac{x}{2^n} = \tan\dfrac{x}{2^n}-\tan\dfrac{x}{2^{n+1}}$. Hence

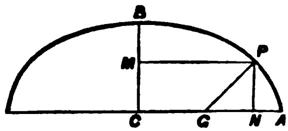
$$S = \left(\tan x-\tan\frac{x}{2}\right)+\left(\tan\frac{x}{2}-\tan\frac{x}{2^2}\right)+\dots+\left(\tan\frac{x}{2^n}-\tan\frac{x}{2^{n+1}}\right)$$

= result stated.

8087. (By the EDITOR.)—If a perfectly smooth oblate hemispheroid be placed on a horizontal plane, with its minor axis ($2b$) vertical and its major axis ($2a$) along the plane, and a particle descend, by its own weight, from rest at the apex of the spheroid; show that (c being the focal distance) the particle will leave the surface when its distance from the plane is

$$\frac{b^{\frac{4}{3}}}{c}\left[(a+c)^{\frac{1}{3}}-(a-c)^{\frac{1}{3}}\right].$$

Solutions by (1) Rev. T. C. SIMMONS, M.A.; (2) G. HEPPEL, M.A., D. BIDDLE, *and others*.



1. Let P be the point of separation; PN, PM perpendiculars on the axes of the vertical ellipse along which the motion begins, PG the normal at P. Put PN $= y$. Then, at P,

$$v^2 = g\rho\cos \mathrm{NPG} \quad ....(1).$$

But $2\rho\cos\mathrm{NPG}$ is twice the chord of curvature at P perpendicular to AC, which, by a well-known formula, is equal to PN . $\mathrm{CD}^2/\mathrm{BC}^2$ or to $y\,(b^4+c^2y^2)/b^4$. Also $v^2=2g\,(b-y)$, whence (1) becomes $c^2y^3+3b^4y-2b^5=0$, a cubic equation whose one real root reduces from CARDAN's formula to the delightfully simple form given in the question.

2. *Otherwise*:—The direction and velocity of the particle at any point of its elliptic path is the same as if it were moving in a parabola touching the ellipse at that point, and having a directrix at a height b above the plane. When the contact becomes one of the second order, this parabola crosses the ellipse and the particle leaves the curve. Now the ellipse being $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$, $\frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}$. Parabola being $(x-h)^2=-2l\,(y-b+\frac{1}{2}l)$, we have $\frac{dy}{dx}=-\frac{x-h}{l}$, $\frac{d^2y}{dx^2}=-\frac{1}{l}$. Equating values of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, we obtain four equations, from which, after eliminating h, l, x, and simplifying, we have $c^2y^3+3b^4y-2b^5=0$, the one real root of which is, by CARDAN's rule, the stated result.

7875. (By S. TEBAY, B.A.)—Find rational values of x and y such that $x^2+cxy+y^2+a$ and $x^2+cxy+y^2-a$ shall be squares.

Solution by the PROPOSER.

Let $x^2+cxy+y^2+a=(y+p)^2$; then $y=\frac{x^2+a-p^2}{2p-cx}$. Substituting in

the second expression, the denominator is $(2p-cx)^2$, and the numerator

$$x^4-2cpx^3+[(c^2+2)\,p^2-2a\,(c^2-1)]\,x^2+2cp\,(3a-p^2)\,x+a^2-6ap^2+p^4.$$

Putting this $=(x^2-cpx+q)^2$, we get

$$2\,[p^2-a\,(c^2-1)-q]\,x^2+2cp\,(3a-p^2+q)\,x+a^2-6ap^2+p^4-q^2=0\,;$$

whence, taking $q=p^2-a\,(c^2-1)$, we find

$$x=\frac{2\,(4-c^2)\,p^2-ac^2\,(c^2-2)-2a}{2cp\,(4-c^2)}.$$

The value of y can be found by substitution.

Otherwise:—Let $x^2+cxy+y^2+a=(m+n)^2$, $x^2+cxy+y^2-a=(m-n)^2$. By addition and subtraction, we have $x^2+cxy+y^2=m^2+n^2$, $a=2mn$. The former of these equations may be written $x^2+(cx+y)\,y=m^2+n^2$. Let $y=\dfrac{q^2}{p^2}(cx+y)$, or $y=\dfrac{cq^2x}{p^2-q^2}$. Take $x=p^2-q^2$, then $y=cq^2$, and $(p^2-q^2)^2+c^2p^2q^2=m^2+n^2$; hence we can take $m=p^2-q^2$, $n=cpq$, $a=2cpq\,(p^2-q^2)$. The roots of the squares are

$$p^2+cpq-q^2 \quad\text{and}\quad p^2-cpq-q^2.$$

The former solution has the merit of finding x, y when a and c are given.

8067. (By B. Hanumanta Rau, M.A.)—If n be any positive integer, prove that $\Pi\equiv(1+\tfrac{1}{1})(1+\tfrac{1}{3})(1+\tfrac{1}{5})(1+\tfrac{1}{7})\,\ldots\ldots\left(1+\dfrac{1}{2n+1}\right)>(2n+3)^{\frac{1}{2}}$.

Solution by A. M. Williams, M.A.; J. O'Regan; *and others.*

We have $\Pi=\dfrac{2}{1}\cdot\dfrac{4}{3}\cdot\dfrac{6}{5}\ldots\ldots\dfrac{2n+2}{2n+1}$; and, since a ratio of greater inequality is diminished by adding the same quantity to both terms of the ratio, $\Pi>\dfrac{3}{2}\cdot\dfrac{5}{4}\cdot\dfrac{7}{6}\ldots\ldots\dfrac{2n+3}{2n+2}$; hence, multiplying, $\Pi^2>2n+3$.

7964. (By Professor Byomakesa Chakravarti, M.A.)—If a man goes in for an examination in which there are four papers, with a maximum of m marks for each paper; prove that the number of ways of getting half-marks on the whole is $\frac{1}{3}\,(m+1)(2m^2+4m+3)$.

Solution by Rev. T. Galliers, M.A.; E. Rutter; *and others.*

The number of ways in question is clearly equal to the coefficient of x^{2m} in the product $(1+x+x^2+\ldots+x^m)^4$ or $(1-x^{m+1})^4(1-x)^{-4}$, which coefficient will be

$$\frac{4\,.\,5\ldots(4+2m-1)}{(2m)!}-4\,\frac{4\,.\,5\ldots[4+(m-1)-1]}{(m-1)!}$$

$$=\frac{4\,.\,5\ldots(2m+3)}{(2m)!}-4\,\frac{4\,.\,5\ldots(m+2)}{(m-1)!}=\frac{(2m+3)!}{1\,.\,2\,.\,3\,(2m)!}-4\,\frac{(m+2)!}{1\,.\,2\,.\,3\,(m-1)!}$$

$$=\frac{(2m+1)(2m+2)(2m+3)}{1\,.\,2\,.\,3}-4\,\frac{m(m+1)(m+2)}{1\,.\,2\,.\,3}=\text{stated result.}$$

7985. (By R. Tucker, M.A.)—"The osculating circles at the points Q, R, S of an ellipse cointersect at P, also on the ellipse; and the points P, Q, R, S are concyclic." The Simson-line for the point P with regard to the triangle QRS is LMN; prove that, if $\lambda\pm\mu=(a\pm b)^3$, as P moves round the ellipse, LMN envelopes the curve

$$(\lambda x)^{\frac{2}{3}}+(\mu y)^{\frac{2}{3}}=(a^2+b^2)^{\frac{4}{3}}.$$

Solution by B. Hanumanta Rau, B.A.; *and* Rev. T. C. Simmons, M.A.

Let 3α be the eccentric angle of P. Then those of Q, R, S are $-\alpha$, $\frac{2}{3}\pi-\alpha$, and $\frac{4}{3}\pi-\alpha$. The equation to RS is

$$-\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=\tfrac{1}{2}.$$

The perpendicular PL on it from P is

$$(x-a\cos 3\alpha)\,a\sin\alpha+(y-b\sin 3\alpha)\,b\cos\alpha=0.$$

The coordinates of L are therefore

$$\frac{a\cos 3\alpha}{m}\left[a^2-(a^2+b^2)\cos 2\alpha\right]\quad\text{and}\quad\frac{b\sin 3\alpha}{m}\left[b^2+(a^2+b^2)\cos 2\alpha\right],$$

where $$m=a^2+b^2-(a^2-b^2)\cos 2\alpha.$$

The coordinates of M and N are obtained by writing $\frac{2}{3}\pi-\alpha$ and $\frac{4}{3}\pi-\alpha$ respectively for 2α within the brackets in the values of the coordinates of L.

The straight line passing through the points (x_1, y_1) and (x_2, y_2) is known to be $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$. Substituting the coordinates of M and N in this equation, the equation to the Simson-line LMN is found to be

$$(a^3+3ab^2)\,x\sin 3\alpha+(b^3+3a^2b)\,y\cos 3\alpha=(a^2+b^2)^2\sin 3\alpha\cos 3\alpha,$$

or $$\frac{\lambda x}{\cos 3\alpha} + \frac{\mu y}{\sin 3\alpha} = (a^2 + b^2)^2 \quad \text{......................(A).}$$

Differentiating with respect to α,

$$\frac{\lambda x \sin 3\alpha}{\cos^3 3\alpha} = \frac{\mu y \cos 3\alpha}{\sin^3 3\alpha} \text{ or } \frac{\sin^3 3\alpha}{\mu y} = \frac{\cos^3 3\alpha}{\lambda x} = \frac{1}{(\lambda x)^{\frac{2}{3}} + (\mu y)^{\frac{2}{3}}}.$$

Substituting these values in (A), we have the stated result.

3189. (By Professor EVANS, M.A.)—Prove that the product of six consecutive integers cannot be a perfect square number.

Solution by the PROPOSER.

Let $P_x = \frac{x\,!}{(x-6)\,!} = (x-5)\,(x-4)\,(x-3)\,(x-2)\,(x-1)\,x$. It is evident that, while $(x-4) < 25$, P_x will contain the prime factor 5 an odd number of times unless $(x-5) = 5m_1$. It is also evident that, while $x < 49$, P_x will contain the prime factor 7 an odd number of times unless $(x-5) = 7m_2 + 1$. Let $x - 5 = 5m_1 = 7m_2 + 1$; then, since x, m_1, and m_2 are integers, we must have $x = 35m + 20$.

When $m = 0$, $x = 20$, in which case P_x is not a square, since it contains the prime factors 19, 17, once each. When $m = 1$, $x = 55$; hence P_∞ cannot be a square while $x < 29$. Let $y = x - \frac{5}{2}$; then we have

$$P_x = (y^2 - \tfrac{1}{4})(y^2 - \tfrac{9}{4})(y^2 - \tfrac{25}{4})$$

$$= y^6 - \frac{35}{4}y^4 + \frac{259}{16}y^2 - \frac{225}{64} = \left(y^3 - \frac{35}{8}y\right)^2 - \frac{189y^2 + 225}{64}.$$

Since P_x cannot be a square while $x < 29$, it cannot be a square while $y < 26\frac{1}{2}$ or while $y < 26$.

Put $y^3 - \frac{35}{8}y = a$, and $\frac{189y^2 + 225}{64} = b$; then $P_x = a^2 - b = a^2\left(1 - \frac{b}{a^2}\right)$;

and, by extracting the square root, we obtain

$$\sqrt{P_x} = a\left\{1 - \frac{1}{2}\cdot\frac{b}{a^2} - \frac{1}{2\,.\,4}\cdot\frac{b^2}{a^4} - \frac{1\,.\,3}{2\,.\,4\,.\,6}\cdot\frac{b^3}{a^6} - \text{\&c.}\right\},$$

or $$\sqrt{P_x} = a - \frac{1}{2}\cdot\frac{b}{a}\left(1 + \frac{1}{4}\,\frac{b}{a^2} + \frac{3}{4\,.\,6}\cdot\frac{b^2}{a^4} + \text{\&c.}\right);$$

where $y > 26$, $a > 17462\frac{1}{4}$, and $b > 1999\frac{53}{64}$ and $\frac{1}{2}\,\frac{b}{a} < \frac{1}{17}$.

But $\left(1 + \frac{1}{4}\,\frac{b}{a^2} + \frac{3}{4\,.\,6}\,\frac{b^2}{a^4} + \text{\&c.}\right)$ is evidently greater than 1 and less

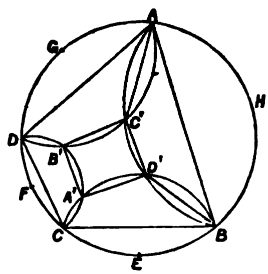
than $\frac{17}{16}$, while $y > 26$; therefore $\sqrt{P_x} = a - \frac{1}{2}\frac{b}{a}\beta$, where β stands for $\left(1 + \frac{1}{4}\frac{b}{a^2} + \frac{3}{4 \cdot 6}\frac{b^2}{a^4} + \&c.\right)$. But $\frac{1}{2}\frac{b}{a} < \frac{1}{17}$ and $\beta < \frac{17}{16}$, therefore $\frac{1}{2}\frac{b}{a}\beta < \frac{1}{16}$. Also $a = \frac{1}{8}(8y^3 - 35y)$; hence, since the fractional part of a is an integral number of eighths, if it has any fractional part, the remainder, found by subtracting a fraction less than $\frac{1}{16}$ from a cannot be an integer. Therefore $\sqrt{P_x}$ cannot be an integer, or

$$\frac{x\,!}{(x-6)\,!} = (x-5)(x-4)(x-3)(x-2)(x-1)\,x$$

cannot be a square; that is, *the product of six consecutive integers cannot be a perfect square.*

8073. (By E. Rutter.)—If ABCD be a quadrilateral inscribed in a circle, prove that the in-centres of the triangles ABC, BCD, CDA, DAB are the vertices of a rectangle.

Solution by "Iris"; Sarah Marks; *and others.*



The in-centres of the triangles ACB, DCB lie in a circle with mid-point E of arc CB as centre, and CE or EB as radius; and so for the other triangles: hence the four in-centres A′, B′, C′, D′ are the four points of intersection of arcs having the mid-points E, F, G, H, of the arcs BC, CD, DA, AB as centres, and EC, FD, GA, HB for radius respectively. We have, therefore,

$$\angle B'C'D' = 2\pi - AC'B' - AC'D'$$
$$= ADB' + ABD' = \tfrac{1}{2}ADC + \tfrac{1}{2}ABC = \tfrac{1}{2}\pi.$$

Similarly for the other angles at D′, A′, B′; hence the figure A′B′C′D′ is a rectangle.

7968. (By Professor Cochez.)—Démontrer la somme

$$\frac{1}{\sin x} + \frac{1}{\sin \frac{x}{2}} + \dots + \frac{1}{\sin \frac{x}{2^n}} = \cot \frac{x}{2^{n+1}} - \cot x.$$

Solution by B. Hanumanta Rau, M.A.; W. J. Greenstreet, B.A.; *and others.*

Now $\dfrac{1}{\sin\dfrac{x}{2^r}} = \sin\left(\dfrac{x}{2^{r+1}} - \dfrac{x}{2^{r+1}}\right) \Big/ \sin\dfrac{x}{2^{r+1}} \cdot \sin\dfrac{x}{2^r} = \cot\dfrac{x}{2^{r+1}} - \cot\dfrac{x}{2^r}$;

hence, substituting the value in the given series, we have

$$\text{the sum} = \left(\cot\frac{x}{2} - \cot x\right) + \left(\cot\frac{x}{2^2} - \cot\frac{x}{2}\right) + \dots\left(\cot\frac{x}{2^{n+1}} - \cot\frac{x}{2^n}\right)$$

$=$ stated result.

8039. (By B. Hanumanta Rau, M.A.)—O is any point within an equilateral triangle ABC; Δ and Δ' are respectively the areas of the triangle ABC and the triangle whose sides are equal to OA, OB, OC. Prove that

$$OA^2 + OB^2 + OC^2 = \frac{4}{\sqrt{3}}(2\Delta - 3\Delta').$$

Solution by H. S. Jones; R. Knowles, B.A.; *and others.*

If a be the side of the triangle, we have

$$\Delta' = \tfrac{1}{2}AO \, . \, BO \sin(AOB - 60^\circ) = \tfrac{1}{4}AO \, . \, BO \sin AOB - \tfrac{1}{4}\sqrt{3}AO \, . \, BO \cos AOB$$

$$= \tfrac{1}{2}\Delta(AOB) - \tfrac{1}{8}\sqrt{3}(AO^2 + BO^2 - a^2) = \tfrac{1}{2}\Delta(AOB) + \tfrac{1}{2}\Delta - \tfrac{1}{8}\sqrt{3}(AO^2 + BO^2);$$

similarly, $\quad \Delta' = \tfrac{1}{2}\Delta(BOC) + \tfrac{1}{2}\Delta - \tfrac{1}{8}\sqrt{3}(BO^2 + CO^2),$

$\quad \Delta' = \tfrac{1}{2}\Delta(AOC) + \tfrac{1}{2}\Delta - \tfrac{1}{8}\sqrt{3}(AO^2 + CO^2);$

therefore, by addition, $AO^2 + BO^2 + CO^2 = \dfrac{4}{\sqrt{3}}(2\Delta - 3\Delta').$

3848. (By J. B. Sanders.)—Find the times in which a fluid contained in a vessel, formed by the revolution of a curve whose equation is $y^4 = a^3x$ about the axis of x, will descend through equal distances h, supposing a small orifice at the vertex, and the axis vertical.

Solution by Professor Evans, M.A.; Professor Matz, M.A.; *and others.*

Let v represent the velocity and k the area of a section of the *vena contracta;* K the area of the surface of the descending fluid at any time t, and x its altitude above the orifice; then, for determining the motion of

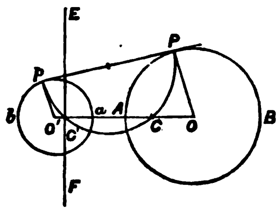
the fluid, we have $v = (2gx)^{\frac{1}{2}}$ and $kvdt = -\mathrm{K}dx$. Therefore, since $\mathrm{K} = \pi y^2 = \pi a^{\frac{3}{2}} x^{\frac{1}{2}}$, we have $k\,(2gx)^{\frac{1}{2}}\,dt = -\pi a^{\frac{3}{2}} x^{\frac{1}{2}} dx$, or

$$dt = -\frac{\pi a^{\frac{3}{2}}}{k\,(2g)^{\frac{1}{2}}}\,dx, \quad \therefore\ t = \frac{\pi a h}{k}\left(\frac{a}{2g}\right)^{\frac{1}{2}},$$

which shows that equal spaces are described in equal times.

7209. (By W. J. C. Sharp, M.A.)—If a circle be described upon a common tangent to two circles as diameter, show that it cuts the line of centres in points which are inverse to each other with respect to both circles. Hence prove that confocal conics cut at right angles.

Solution by the Proposer.



If O, O′ be the centres of the given circles, Pp the common tangent, and C, C′ the points in which the circle on Pp as a diameter cuts OO′: since OP and O′p touch this last circle, $\mathrm{OP}^2 = \mathrm{OC}\,.\,\mathrm{O'C'}$ and $\mathrm{O'}p^2 = \mathrm{O'C}\,.\,\mathrm{O'C'}$, which proves the first part. Hence EP the line at right angles to OO′ through C′ is the polar of C with respect to each of the two first circles. If these be reciprocated with respect to an auxiliary circle, centre C; C is a focus of each of the conics thus obtained, also C and its polars EF with respect to each circle reciprocated correspond to infinity and its polar (the centre) with respect to the corresponding conic. The conics are therefore concentric, and consequently confocal. Also the line Pp corresponds to a point of intersection, and, since CP and Cp are at right angles, the tangents at this point are at right angles. [For another Solution, see Salmon's *Conics*, 5th Ed., Arts. 188, 316.]

4473. (By Professor Evans, M.A.)—Show that the product of seven consecutive integers cannot be the square of a commensurable number.

Solution by the Proposer.

Let x be the smallest of the seven numbers; then

$$x\,(x+1)\,(x+2)\,(x+3)\,(x+4)\,(x+5)\,(x+6) = y^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is the condition whose impossibility is to be shown.

It is evident that the only prime numbers common to any two of the seven factors x, $(x+1)\ldots(x+6)$, are 2, 3, 5, since these are the only prime numbers in the series 1, 2, 3, 4, 5, 6 which contains all such divisors.

No one of the seven factors under consideration can contain as divisors all the numbers 2, 3, 5; and any one of these seven factors that does not contain 2, 3, or 5 as divisor, is an exact square, if condition (1) is true, as may readily be shown (*Nouvelles Annales, First Series*, Vol. XIX., p. 38).

We will now show that condition (1) leads to this absurdity, *that no one of seven consecutive integers is divisible by* 6. If $x = 6m$, $x+6 = 6(m+1)$, and neither m nor $(m+1)$ is divisible by 5. If $x = 6m$, neither $x+1 = 6(m+1) - 5$, nor $(x+5) = 6(m+1) - 1$ is divisible by any one of the three factors 2, 3, 5; and therefore $(x+1)$ and $(x+5)$ are both squares, and we have two integral squares whose difference is only four units, which is impossible; therefore neither x nor $x+6$ is divisible by 6.

It remains to consider the intermediate factors $(x+1)$, $(x+2)$, $(x+3)$, $(x+4)$, $(x+5)$. If one of these, $(x+2)$ for example, is a multiple of 6, the two factors $(x+1)$, $(x+3)$, which include $(x+2)$ between them, will be prime to 6; and, as they cannot both be multiples of 5, one of them, not admitting any one of the factors 2, 3, 5, will be a square, a^2: the other $a^2 \pm 2$, not being a square, ought to be a multiple of 5, or $a^2 \pm 2$ ought to equal $5n$. This condition requires $a^2 = 5n \pm 2$, which is clearly impossible, since $5n \pm 1$ is the formula for all squares that are not multiples of 5. Hence the impossibility of condition (1) is fully established.

8166. (By Rev. T. C. SIMMONS, M.A. Suggested by Quest. 7713.)—Three or more coins are thrown at random on a rectangular table. Find the chance that they will all lie on a random line drawn parallel to the edge of the table.

7713. (By the EDITOR.)—If a florin, a shilling, and a sixpence be thrown at random on a rectangular table, find, by a general solution, the probability that the three coins will all lie on a line parallel to the edge of the table.

Solution by D. BIDDLE.

(8166.) Let a, b = edges of table, c_1, c_2, c_3, &c. = diameter of coins. Taking, in this instance, the number of possible lines in the two directions as apportioned according to the lengths of the sides to which they are perpendicular, the chance that the random line will be parallel to a, is $\frac{b}{a+b}$, and that it will be parallel to b, is $\frac{a}{a+b}$.

Let us then suppose infinitesimal (equal) counters instead of coins to be thrown, and the diameter of each to be dm. The chance that anyone in particular will be within proper range of the random line, is

$$\left(a\frac{b}{a+b} + b\frac{a}{a+b}\right) dm / ab = \frac{2dm}{a+b},$$

and the chance that all will be within proper range is $\left(\frac{2dm}{a+b}\right)^n$.

Next, let us consider the average range of each coin. Taking c_1 as an example, we find that, if the random line be parallel to the b-edge, the coin has its full range over $(a-c_1)\,b\,/\,ab$ of the table; but, for a distance $(=\frac{1}{2}c_1)$ within each b-edge, the range is curtailed, varying from 1 to $\frac{1}{2}$, the mean being $\frac{3}{4}$. This gives $[(a-c_1)\,b+\frac{3}{4}bc_1]\,c_1\,/\,ab$ as the average range when the line is in that direction. Similarly, when the line is parallel to a, the average range is $[(b-c_1)\,a+\frac{3}{4}ac_1]\,c_1\,/\,ab$. Consequently, taking the chance as to direction as before, we have

$$\frac{b}{a+b}\left[(b-c_1)\,a+\tfrac{3}{4}ac_1\right]c_1\,/\,ab+\frac{a}{a+b}\left[(a-c_1)\,b+\tfrac{3}{4}bc_1\right]c_1\,/\,ab,$$

$$=\left[1-\frac{c_1}{2\,(a+b)}\right]c_1,$$

as the working range of this coin. And by substituting c_2, c_3, &c. for c_1, we readily find the average ranges of the other coins.

Moreover, the respective ranges of the coins, in any one direction, are to those of the counters as $c_1 : dm$, $c_2 : dm$, $c_3 : dm$, &c. But

$$\left(\frac{2dm}{a+b}\right)^n\cdot\frac{c_1c_2\ldots c_n}{(dm)^n}=\frac{2^n\,.\,c_1\,.\,c_2\ldots c_n}{(a+b)^n}.$$

Therefore, the probability that n coins will all lie on the random line, is

$$\frac{2^n\,.\,c_1\,.\,c_2\ldots c_n}{(a+b)^n}\cdot\left[1-\frac{c_1}{2\,(a+b)}\right]\cdot\left[1-\frac{c_2}{2\,(a+b)}\right]\ldots\left[1-\frac{c_n}{2\,(a+b)}\right],$$

or $$\mathrm{P}_n=\frac{c_1\,.\,c_2\ldots c_n}{(a+b)^{2n}}\left[2\,(a+b)-c_1\right]\left[2\,(a+b)-c_2\right]\ldots\left[2\,(a+b)-c_n\right].$$

Thus, if $c_1\,(=\frac{47}{40})$, $c_2\,(=\frac{37}{40})$, $c_3\,(=\frac{30}{40})$ be the respective diameters of a florin, a shilling, and a sixpence, and if $a=40$, $b=30$, then $\mathrm{P}_3=\frac{1}{87500}$ nearly.

(7713.) This question, though it suggested that above considered, differs materially from it. In 7713, the line is determined by the position of the coins; in 8166, it is drawn separately, though at random. In 7713, the line referred to is one of numerous possible lines within a rectangular space varying in width from zero to the diameter of the smallest coin; in 8166, the line indicated is single and distinct. In 7713, p_1 is unity, being certain; in 8166, P_1 is evidently a fraction. The difference is much the same as between the respective probabilities that three assigned persons will occupy a compartment specified at random, and that they will on the other hand together occupy *some* one compartment, in a given train. The probability as to direction of the line is also different. In 7713, it is always more probable that the coins should arrange themselves in a line parallel to the longer edge of the table; in 8166, the probability in regard to the random line is the reverse, as we have seen.

Treating Quest. 7713, as we have treated Quest. 8166, and at first taking infinitesimal counters, we find that

$$p_1=1,\quad p_2=2dm\,(a+b-2dm)\,/\,ab,\quad p_3=p_2\,.\,\tfrac{3}{2}dm\left(\frac{a^2}{a+b}+\frac{b^2}{a+b}\right)\Big/\,ab,$$

and $$p_n=\left(\frac{2\,.\,3\,.\,4\ldots n}{1\,.\,2\,.\,3\ldots(n-1)}\right)dm^{n-1}\,(a+b-2dm)$$

$$\times\left\{\left(\frac{a^2}{a+b}\right)^{n-2}+\left(\frac{b^2}{a+b}\right)^{n-2}\right\}\Big/\,(ab)^{n-1}.$$

Here, the subtraction of $2dm$ from $a+b$, in p_2, is due to crossing of representative areas; and $\frac{2}{1}$, $\frac{3}{2}$, $\frac{4}{3}$, &c., represent the mean width of area which the successive counters (beginning with the second) are allowed to range over.

Now p_3 may be expanded as follows :—

$$\left.\begin{aligned} p_3 &= (dm_1+dm_2)(a+b-dm_1-dm_2)(\tfrac{1}{2}dm_2+dm_3) \\ &= (dm_2+dm_3)(a+b-dm_2-dm_3)(\tfrac{1}{2}dm_3+dm_1) \\ &= (dm_3+dm_1)(a+b-dm_3-dm_1)(\tfrac{1}{2}dm_1+dm_2) \end{aligned}\right\}$$

$$\times \left\{ \left(\frac{a^2}{a+b}\right) + \left(\frac{b^2}{a+b}\right) \right\} \Big/ (ab)^2.$$

But when we substitute for dm_1, dm_2, dm_3 (which are equal) the average ranges of the coins (which are unequal), viz.,

$$c_1 - \frac{c_1^2}{2(a+b)}, \quad c_2 - \frac{c_2^2}{2(a+b)}, \quad c_3 - \frac{c_3^2}{2(a+b)},$$

we must take the mean of the factors indicated above. This gives, for Quest. 7713, taking the same numerical values as before,

$$p_3 = \frac{15{,}860{,}956{,}802{,}851{,}136{,}668}{3{,}625{,}388{,}605{,}440{,}000{,}000{,}000} = \frac{1}{229} \text{ nearly.}$$

Now, although it might on principle be preferable to take, in the above instances, the product of the several single probabilities for every parallel, and derive an average thence; and although in many cases there is a considerable difference between what we may call the *average of products* (found by the Integral Calculus) and the *product of averages* (as given above); yet, in the present case, the difference can be but slight, whilst by the method adopted great difficulties are obviated.

[*Otherwise* :—Supposing a rectangular table, of length l, to have a rim running along its sides, and n coins to be thrown at random, of which the diameters, in increasing order of magnitude, are $a_1, a_2, \dots a_n$; then, denoting by p_n the probability that the coins will all lie on a random line drawn parallel to the breadth of the table, and putting $l_1 = l - a_1$, $l_2 = l - a_2$, it is clear that the probability of a parallel lying between the distances x, $x+dx$ from one end of the table is $dx : l$, and the probability that the coin (a_1) will rest on this parallel is the ratio that the range of the coin's centre across the parallel bears to its entire linear range along the table; that is to say, $a_1 : l_1$, when $x > a_1$, and $x : l_1$, when $x < a_1$. We thus see that the total probability required will be given by taking, as in the summation hereunder, the several partial probabilities as x increases from 0 to $\frac{1}{2}l$, and doubling their sum to allow for the other half of the table. We have, therefore, for the several probabilities,

$$p_2 = \frac{2}{ll_1 l_2}\left\{\int_0^{a_1} x^2 dx + \int_{a_1}^{a_2} a_1 x\, dx + \int_{a_2}^{\frac{1}{2}l} a_1 a_2\, dx\right\}$$

$$= \frac{2}{ll_1 l_2}\left\{\tfrac{1}{3}a_1^3 + \tfrac{1}{2}a_1(a_2^2 - a_1^2) + a_1 a_2(\tfrac{1}{2}l - a_2)\right\} = \frac{a_1}{ll_1 l_2}(l_2 a_2 - \tfrac{1}{3}a_1^2);$$

$$p_3 = \frac{2}{l l_1 l_2 l_3} \left\{ \int_0^{a_1} x^3 dx + \int_{a_1}^{a_2} a_1 x^2 dx + \int_{a_2}^{a_3} a_1 a_2 x dx + \int_{a_3}^{\frac{1}{2}l} a_1 a_2 a_3 \, dx \right\}$$

$$= \frac{2}{l l_1 l_2 l_3} \left\{ \tfrac{1}{4} a_1^4 + \tfrac{1}{3} a_1 (a_2^3 - a_1^3) + \tfrac{1}{2} a_1 a_2 (a_3^2 - a_2^2) + a_1 a_2 a_3 (\tfrac{1}{2} l - a_3) \right\}$$

$$= \frac{a_1}{l l_1 l_2 l_3} (l_3 a_2 a_3 - \tfrac{1}{3} a_2^3 - \tfrac{1}{6} a_1^3);$$

$$p_n = \frac{2}{l l_1 l_2 \dots l_n} \left\{ \int_0^{a_1} x^n dx + \int_{a_1}^{a_2} a_1 x^{n-1} dx + \int_{a_2}^{a_3} a_1 a_2 x^{n-2} dx + \dots \right.$$

$$\left. \dots + \int_{a_{n-1}}^{a_n} a_1 a_2 a_3 \dots a_{n-1} x \, dx + \int_{a_n}^{\frac{1}{2}l} a_1 a_2 a_3 \dots a_n \, dx \right\}$$

$$= \frac{2}{l l_1 l_2 \dots l_n} \left\{ \frac{1}{n+1} a_1^{n+1} + \frac{1}{n} a_1 (a_2^n - a_1^n) + \frac{1}{n-1} a_1 a_2 (a_3^{n-1} - a_2^{n-1}) \right.$$

$$\left. + \frac{1}{n-2} a_1 a_2 a_3 (a_4^{n-2} - a_3^{n-2}) + \dots + a_1 a_2 a_3 \dots a_n (\tfrac{1}{2} l - a_n) \right\}$$

$$= \frac{2}{l l_1 l_2 \dots l_n} \left\{ - \frac{a_1^{n+1}}{(n+1) n} - \frac{a_1 a_2^n}{n (n-1)} - \frac{a_1 a_2 a_3^{n-1}}{(n-1)(n-2)} - \dots + \tfrac{1}{2} l a_1 a_2 a_3 \dots a_n \right\}$$

Taking a sixpence, a shilling, and a florin to have diameters ·7625, ·925, and 1·175, and the length of the table to be 40 inches, we have, when the sixpence and shilling only are tossed,

$$p_2 = \frac{\cdot 7625 \, [39 \cdot 075 \times \cdot 925 - \frac{1}{3} (\cdot 7625)^2]}{1563 \times 39 \cdot 2375} = \cdot 000440423,$$

and when all three coins are tossed

$$p_3 = \frac{\cdot 7625 \, [38 \cdot 825 \times \cdot 925 \times 1 \cdot 175 - \frac{1}{3} (\cdot 925)^3 - \frac{1}{6} (\cdot 7625)^3]}{40 \times 39 \cdot 2375 \times 39 \cdot 075 \times 38 \cdot 825} = \cdot 000013405.]$$

Note on Questions 8166 & 7713. *By the* Editor.

These questions, with their solutions, have led to much discussion and controversy, of which, so far as concerned with the *principles* of the Theory of Probability, we here present a synopsis.

Mr. Simmons thinks that Quest. 8166 cannot be correctly solved without the aid of the Integral Calculus, and objects to "the assumption in Mr. Biddle's solution, that if the chance of a coin A falling on a random line is p, and the chance of a coin B falling on a random line is q, then the chance of A and B falling together on a random line is pq. For the same argument would show that, if the chance of a passenger A being in a compartment selected at random in a railway train were p, and the chance of B being in a random compartment were q, then the chance of

their being together in a random compartment would be pq, a conclusion which we will test in the following particular case. Suppose that A always travels first-class, B always third-class, and that the train consists of five of the former compartments, and ten of the latter. The chance that A will be in a random compartment is here plainly $\frac{1}{15}$, the chance that B will be in a random compartment is also $\frac{1}{15}$, while the chance of their being *together* in a random compartment is *not* $\frac{1}{225}$, *but* 0."

To this Mr. BIDDLE replies that "the two cases are not analogous. In the case of the passengers, as given by Mr. SIMMONS, the fact of their never travelling by the same class precludes all probability of their travelling together, and prior knowledge of this fact on our part would preclude all possibility of our entertaining the question. By a parity of reasoning, we are driven to the conclusion that a complete knowledge of all the circumstances of any case whatever must of necessity obviate all question of probability, of which uncertainty is one of the main props. But, in the case before us, the coins and counters are none of them precluded, by any stated condition, from lying on any part of the table; and we assume with strict impartiality that the centre of each may lie anywhere on the surface. The question, therefore, as regards the separate chance of each coin, becomes one of simple ratio between areas—that of the whole table on the one hand, and that which is defined by tangents to the coin on the other, the mean being taken for all positions. But we further assume that each coin or counter takes up its position independently, that its lying where it does is an independent event, so far as the other coins or counters are concerned (although their positions may coincide); and it is on this ground that, according to the well-known formula, we take the product of the separate chances to determine the probability as regards all."

Mr. SIMMONS rejoins that "the prior knowledge in question does not affect the argument, for *any* hypothesis which made the chance of the travelling together zero, ought, if Mr. BIDDLE's principle had been correct, to have made one of the factors p or q zero likewise. However, if the above instance is not sufficiently analogous, another can easily be given. Suppose this time that A's only bias in the choice of compartments is a dislike of the one nearest the engine, and that he is twice as likely to travel in any one of the other fourteen compartments. Suppose, moreover, that B's bias is exactly the same as A's. Here the chance that A will be in a random compartment is $\frac{1}{15}$, and the same chance holds for B; whereas the chance of their being *together* in a random compartment is not $\frac{1}{225}$, but $\frac{14}{15}\left(\frac{2}{29}\right)^2+\frac{1}{15}\left(\frac{1}{29}\right)^2$. The case of the coins in (8166) is exactly similar to this, for any coin is more likely to lie on a given line drawn across the middle of the table than on a line drawn near the edge. I do not hesitate then to repeat that, even if all the coins were of equal diameters, Mr. BIDDLE's solution would still be incorrect. A formula applicable only to independent *simple* events has been applied to coincidences, which are *compound* events. It is not true, as a rule, that the chance of the coinciding of two coincidences is the product of the chances of the coincidences separately. If p denote the chance that a coin A, and q the chance that a coin B, will fall on a random line, let us see what this involves. Throw down A, and draw a random line; then throw down B, and draw another random line. It is clear that pq denotes the chance

that A will lie on the first line and B on the second. The error lies in assuming that pq denotes the chance *that A and B will both lie on the first drawn line;* whereas the latter chance ought to be denoted by the product of p and the chance of B coinciding with a *line on which A is known to be situated.* Nor is the matter mended by the observation that the difference between this solution and a correct one 'can be but slight.' The *numerical* difference of the results may be but slight when the dimensions of the table are very large compared with the diameters of the coins; but even then an approximate result is of little value unless we have some idea to what extent (to how many decimal places, for instance) it is reliable. The EDITOR's solution of Question 8166, by the Integral Calculus, appended in the Editorial bracket, is, I think, the correct one; and I believe the true connexion between (7713) and (8166) to lie in the fact that the latter is a particular case of the former. If (7713) be solved for $n+1$ coins, and the diameter of the smallest coin be then made equal to zero, we ought apparently to obtain the solution of (8166) for n coins; and on this ground I cannot help thinking that Mr. BIDDLE's solution of (7713) is incorrect also, even for the supposed case of the infinitesimal counters."

Mr. BIDDLE further replies:—"We are here dealing not with many random lines, but with *one* random line, and the respective probabilities of its coinciding with each and with all of several discs. As to *knowing* that one or other disc will be situated in a particular position, this is clearly impossible; but if the position of the line were given, and p, q, r were the several probabilities of three discs coinciding with it, pqr would accurately represent the probability of all three lying on it. And even in regard to a random line, if these were the only data we had (that is, if we did not know the diameters of the discs, or the dimensions of the table, but simply that p, q, r were the several probabilities taken singly), we should be justified in giving pqr as the nearest approach to the required probability which our knowledge of the case would admit of. But having already admitted that, in cases of this kind, the *average of products* is (as a rule) preferable to the *product of averages*, and having departed from that rule simply on account of the exigencies of (7713), I need now only point out the very slight difference between the results obtained by the two methods in (8166). The solution within brackets gives the probability in the case of a line drawn parallel to one edge only. Taking the breadth of the table as 30 inches, the probability, when the line is known to be parallel to the length, would, by the same method, be $\cdot 0000321464$.

Therefore $$P = \tfrac{1}{70}(\cdot 0000321464 \times 30 + \cdot 000013405 \times 40) = \cdot 0000200084.$$

By the other method, taking the same values for the coins, instead of those which I gave, we obtain,

$$P = \frac{1}{70^6}(\cdot 7625 \times \cdot 925 \times 1\cdot 175)(140 - \cdot 7625)(140 - \cdot 925)(140 - 1\cdot 175) = \cdot 0000189367.$$

But that is for Question 8166 as it stands. The solution within brackets supposes the table to have a *rim* running round it. To correspond with this, we must alter our formulæ, so that for c_1 the chance will be

$$\frac{a}{a+b}\left[(a-2c_1)+\tfrac{1}{2}c_1\right]c_1 \,/\, (a-c_1)^2 + \frac{b}{a+b}\left[(b-2c_1)+\tfrac{1}{2}c_1\right]c_1 \,/\, (b-c_1)^2,$$

$$\text{and } P_3 = \frac{c_1 \cdot c_2 \cdot c_3}{(a+b)^3}\left[\frac{40\,(38\cdot8562)}{(39\cdot2375)^2}+\frac{30\,(28\cdot8562)}{(29\cdot2375)^2}\right]$$

$$\times\left[\frac{40\,(38\cdot612)}{(39\cdot075)^2}+\frac{30\,(28\cdot612)}{(29\cdot075)^2}\right]\cdot\left[\frac{40\,(38\cdot237)}{(38\cdot825)^2}+\frac{30\,(28\cdot237)}{(28\cdot825)^2}\right]=\cdot000020145,$$

agreeing to 6 places of decimals with the solution within brackets, when the latter is fully worked out (as shown on p. 74). What is the slight difference apparent here, compared with that which arises, when on the one hand we apportion the probabilities for the two edges in the manner given above, and on the other hand take the simple mean between the probabilities, in order to arrive at the final result? By many persons, however, the latter treatment is considered the right one. It is not well to strain at gnats, whilst we swallow camels.

"As to the theory propounded by Mr. Simmons, that Question 8166 is a particular case of Question 7713, and that a zero-coin in the latter is sufficient to determine the random line in the former, it would be of doubtful soundness, even were the table always square. But, where the table is oblong, it requires us to assume that a random line will be drawn parallel to the longer edge with greater (instead of less, or even equal) probability; for there can be no doubt that $n+1$ coins, tossed as in Question 7713, will arrange themselves in a line parallel to this edge with greater frequency than in the other direction."

Mr. Simmons makes the following final rejoinder:—"The 'camel' to which Mr. Biddle alludes, and which he seems to consider a matter of great importance, is whether the word 'edge' in the Questions is, or is not, to be interpreted 'given edge'; this, being purely a matter of opinion, I shall certainly not waste time in discussing. The unnecessarily laborious calculation given in the last paragraph only points out what was already evident *à priori*; viz., that, when the table is very large compared with the coins, there cannot be much difference between the two numerical results. The point at issue, by no means a 'gnat,' but involving an important principle, will be seen more clearly by taking the table to be square (length $= l$), without a rim, and the diameters of all the coins to be equal ($= a$). Mr. Biddle's formula would then give the result for n coins as $P_n = (2l)^{-2n}(4al-a^2)^n$. If in the bracketed solution we make the necessary allowance for absence of rim, the method would give

$$p_n = \frac{2}{l^{n+1}}\left\{\int_0^{\frac{1}{2}a}\left(\tfrac{1}{2}a+x\right)^n dx + \int_{\frac{1}{2}a}^{\frac{1}{2}l} a^n\,dx\right\} = \frac{a^n}{l^n} - \frac{a^{n+1}}{(n+1)\,l^{n+1}}\left(n-1+\frac{1}{2^n}\right).$$

Now put $l = 6$, $a = 2$; we then obtain for different values of n these corresponding values for P_n and p_n,

n	1		3	4	5	6	7
P_n	$\frac{11}{36}$	·0933	·0285	·00871	·00266	·00081	·00025
p_n	$\frac{11}{36}$	·0957	·0305	·00982	·00319	·00104	·00034

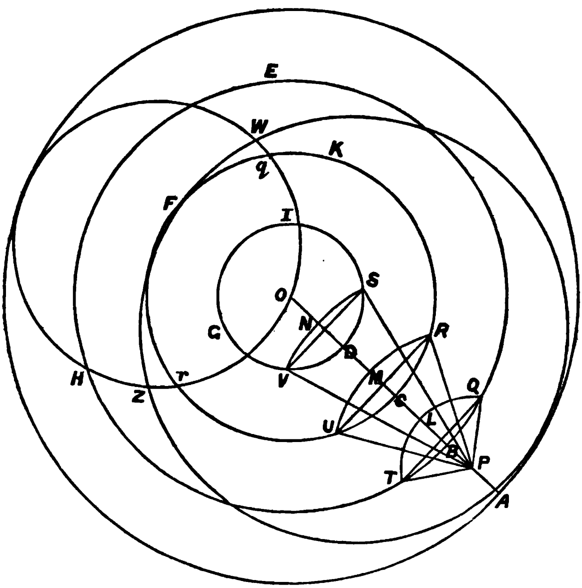
The discrepancy between the two results being in this case so large, and increasing with every additional disc, Mr. BIDDLE will agree with me at any rate in this, that *one* of the two solutions must be *entirely* erroneous.

Questions 8320, 8330 have been proposed with the object of still further elucidating the point at issue, which is also involved in Question 8285.

8100. (By D. BIDDLE.)—Three circles, of diameter 1, 2, 3, are wholly contained by a fourth circle, of diameter 4, but are placed at random in it. Find the respective probabilities that a point, taken at random in the larger circle, shall lie in 0, 1, 2, 3 of the smaller ones.

Solution by the PROPOSER.

This is a case of unequal distribution. If the areas of the three smaller circles were represented by movable sectors of the larger one and concentric therewith, every point in the latter would have the same chance of



being covered by the several sectors, and P_3 would be $\frac{9}{18} \times \frac{4}{18} \times \frac{1}{18}$. But that this does not agree with the probability in the present case, may be seen by taking points in various positions throughout the larger circle, and finding the mean chance of them all. Let a, b, c be (the radii of) the three smaller circles in ascending order. When concentric with the larger circle, they will extend to IGD, KFC, EHB respectively; but the range of their centres will be exactly inverse.

Let OA be a radius of the larger circle, and P a point on it, also $PQ = a$, $PR = b$, $PS = c$. Then QLTB / EHB, RMUC / KFC, and SNVD / IGD will be the probabilities that P will be covered by the smaller circles respectively; and, if with radius OP a circle were described, the probabilities would be the same for every point on its circumference. Moreover, QLTB and SNVD are equal in area, being formed by equal radii from opposite extremities of the same straight line. Consequently the probability that P will be covered by a, is one-ninth of that by c. Also, when P is within OC, the probability in regard to c reaches 1 (or certainty), and in regard to a is $\frac{1}{9}$. The probability in regard to b increases continuously from A to O.

Let OA = unity, and OP = x, then we have

$$\cos QPL = \frac{2x^2-1}{x}, \quad \cos SPO = \frac{2x^2+1}{3x},$$

$$\tfrac{1}{18}\left\{\cos^{-1}\left(\frac{2x^2-1}{x}\right) - \frac{2x^2-1}{x}\left[1-\left(\frac{2x^2-1}{x}\right)^2\right]^{\frac{1}{2}}\right\}$$

$$+\tfrac{9}{18}\left\{\cos^{-1}\left(\frac{2x^2+1}{3x}\right) - \frac{2x^2+1}{3x}\left[1-\left(\frac{2x^2+1}{3x}\right)^2\right]^{\frac{1}{2}}\right\}$$

$$= QLTB = SNVD,$$

Similarly, $\cos RPO = x$, and $\sin RPO = (1-x^2)^{\frac{1}{2}}$,

whence $$RMUC = \tfrac{1}{2}\left[\cos^{-1}x - x(1-x^2)^{\frac{1}{2}}\right].$$

Now, taken singly, the probabilities that a point taken at random in the larger circle shall lie in the several smaller circles, are as the respective areas of those circles, viz., $\frac{9}{18}$, $\frac{4}{18}$, $\frac{1}{18}$. But the case is greatly altered when we consider the different combinations. Thus, to take an instance which is immediately apparent, the area covered by b and c in combination is never less than the double-convex space OWFZ, whose area, in proportion to that of the larger circle, is ·10695, and of this minimum space the portion OqFr (= ·09775) is within the circle FKC, to all points in which the fullest probability is attached, viz., $\frac{IGD}{EHB} = \frac{1}{9}$, as that of its being covered by the smallest of the three movable circles. Consequently the lowest estimate that can be formed of P_3 must exceed $\frac{1}{9}$(·09775) = ·01086, and even this is greater than the supposed average $\frac{9}{18} \times \frac{4}{18} \times \frac{1}{18}$ = ·008789. The following tables give approximate values for integral values of x from 0 to 20:—

Probabilities that a point P, at distance x, from the centre of the larger circle (r = 20), shall be covered by the several smaller circles.

x	By a.	By b.	By c.
20	0	0	0
19	·0051014	·013320	·082082
18	·0140713	·037385	·126642
17	·0252770	·068146	·227490
16	·0379610	·104090	·341650
15	·0515860	·144294	·464270
14	·0656710	·188120	·591040
13	·0796820	·235075	·717140
12	·0929360	·284753	·836420
11	·1042240	·336830	·938020
10	·1111111	·391000	1·000000
9	·1111111	·447013	1·000000
8	·1111111	·504630	1·000000
7	·1111111	·563640	1·000000
6	·1111111	·623835	1·000000
5	·1111111	·685040	1·000000
4	·1111111	·747060	1·000000
3	·1111111	·809730	1·000000
2	·1111111	·872890	1·000000
1	·1111111	·936364	1·000000
0	·1111111	1·000000	1·000000

Probabilities as to 3, 2, 1, 0 *circles covering a point at distance x from centre of larger circle* (*r* = 20),

x	3	2	1	0
20	0	0	0	1·
19	·000006	·001564	·097360	·901070
18	·000066	·006843	·164211	·828880
17	·000392	·021800	·276139	·701669
16	·001350	·048432	·382776	·567442
15	·003456	·088018	·473754	·434772
14	·007302	·140451	·542026	·310221
13	·013433	·204159	·583286	·199122
12	·022135	·275968	·595775	·106122
11	·032930	·350030	·580215	·036825
10	·043445	·415237	·541318	·000000
9	·049668	·458788	·491544	·000000
8	·056067	·503574	·440359	·000000
7	·062626	·549498	·387876	·000000
6	·069315	·596316	·334369	·000000
5	·076116	·643921	·279963	·000000
4	·083007	·692159	·224834	·000000
3	·089970	·740901	·169129	·000000
2	·096988	·790023	·112989	·000000
1	·104041	·839396	·056563	·000000
0	·111111	·888888	·000000	·000000
Average after mult^n. by x.	·0211952 $=P_3$	·2131705 $=P_2$	·3457324 $=P_1$	·4199019 $=P_0$

[On the foregoing solution Mr. SIMMONS makes the following comments : "Mr. BIDDLE's solution is so admirable that I am most unwilling to suggest that it really belongs to a different question, which might be worded somewhat as follows :—'A point being taken at random on the perimeter of one of 20 concentric equidistant circles, its chance of lying on any perimeter being proportional to the length thereof : find its chance of lying within one, two, or three other circles, &c.' The correct solution of Quest. 8100 cannot be effected without integration, and the requisite integration is of such a formidable character as to appear almost hopeless." To this criticism Mr. BIDDLE replies that his solution "does not profess to be more than approximate. But this it is, and, until a more exact result can be attained by the Integral Calculus, it may surely be allowed to stand."]

8130. (By the EDITOR.)—If a, b, c be the sides of a triangle,

$$s = \tfrac{1}{2}(a+b+c), \quad s_1 = s-a, \quad s_2 = s-b, \quad s_3 = s-c,$$

d the diameter of a circle that touches externally the circles on a, b, c as diameters, and $f(s_1) \equiv \left(\frac{d-s_1}{s_1}\right)^{\frac{1}{2}}$, &c.; prove that

$$f(s_1) + f(s_2) + f(s_3) = \{f(s)\}^{-1}.$$

Solution by Rev. J. J. Milne, M.A.; Rev. T. C. Simmons, M.A.; *and others.*

Let D, E, F be the mid-points of BC, CA, AB, O the centre of the circle, and ρ its radius [the figure can be easily imagined]; then we have

$$\rho = \mathrm{BD} + \mathrm{OD} = \mathrm{CE} + \mathrm{OE} = \mathrm{AF} + \mathrm{OF},$$

or

$$d = a + 2\mathrm{OD} = b + 2\mathrm{OE} = c + 2\mathrm{OF}.$$

Now

$$\Delta\mathrm{ABC} = 4\Delta\mathrm{DEF} = 4\Delta\,(\mathrm{OEF} + \mathrm{OED} + \mathrm{ODF}),$$

therefore

$$(ss_1s_2s_3)^{\frac{1}{2}} = [(\mathrm{OE} + \mathrm{OF} - \mathrm{EF})(\mathrm{OF} + \mathrm{EF} - \mathrm{OE})(\mathrm{EF} + \mathrm{OE} - \mathrm{OF}) \times (\mathrm{OE} + \mathrm{OF} + \mathrm{EF})]^{\frac{1}{2}} + [\ldots] + [\ldots]$$

$$= \left\{\left[\tfrac{1}{2}(d-b) + \tfrac{1}{2}(d-c) - \tfrac{1}{2}a\right]\left[\tfrac{1}{2}(d-c) + \tfrac{1}{2}a - \tfrac{1}{2}(d-b)\right] \times \left[\tfrac{1}{2}a + \tfrac{1}{2}(d-b) - \tfrac{1}{2}(d-c)\right]\left[\tfrac{1}{2}(d-b) + \tfrac{1}{2}(d-c) - \tfrac{1}{2}a\right]\right\}^{\frac{1}{2}}$$

$$+ [\ldots] + [\ldots] = [(d-s)(s_3)(s_2)(d-s_1)]^{\frac{1}{2}} + [\ldots]^{\frac{1}{2}} + [\ldots]^{\frac{1}{2}};$$

whence, dividing by $[s_1s_2s_3(d-s)]^{\frac{1}{2}}$, the stated result follows.

8089. (By S. Roberts, M.A.)—Let A, B, C, D be the apices of a given tetrahedron. Take three spheres (1), (2), (3), such that (1) touches the face ABD at B and passes through C; (2) touches the face ABC at C and passes through D; and (3) touches the face ACD at D and passes through B. Then the three spheres intersect in a Brocard-point of the base BCD, and in a point on the circumscribing sphere of the tetrahedron. The latter point and the corresponding point next mentioned are on one and the same sphere passing through the Brocard-circle of the base BCD.

Taking the reverse order of faces and opposite vertices, we have a corresponding point; and, taking as base all the faces in succession, we get four pairs of corresponding points on the circumscribing sphere. Show this. [The key to this is the theorem that, if we take an arbitrary point on each of the edges of a tetrahedron and describe spheres each through a vertex and three of the arbitrary points on adjacent edges, the four spheres intersect in a point. The construction in the question is a particular case.]

Solution by the Proposer.

If the circumscribing sphere is denoted by

$$\mathrm{M} = \mathrm{M}_{12}\lambda_1\lambda_2 + \mathrm{M}_{13}\lambda_1\lambda_3 + \mathrm{M}_{14}\lambda_1\lambda_4 + \mathrm{M}_{23}\lambda_2\lambda_3 + \mathrm{M}_{24}\lambda_2\lambda_4 + \mathrm{M}_{34}\lambda_3\lambda_4 = 0,$$

and the plane at infinity by $\mathrm{N} = \mathrm{N}_1\lambda_1 + \mathrm{N}_2\lambda_2 + \mathrm{N}_3\lambda_3 + \mathrm{N}_4\lambda_4 = 0$, the equations of the three spheres first mentioned are found to be

$$\mathrm{M} - \frac{\mathrm{N}}{\mathrm{N}_2}(\mathrm{M}_{12}\lambda_1 + \mathrm{M}_{24}\lambda_4) = 0, \quad \mathrm{M} - \frac{\mathrm{N}}{\mathrm{N}_3}(\mathrm{M}_{13}\lambda_1 + \mathrm{M}_{23}\lambda_2) = 0,$$

$$\mathrm{M} - \frac{\mathrm{N}}{\mathrm{N}_4}(\mathrm{M}_{14}\lambda_1 + \mathrm{M}_{34}\lambda_3) = 0,$$

which intersect the circumscribing sphere in the point

$$\mathrm{M}_{12}\lambda_1 + \mathrm{M}_{24}\lambda_4 = \mathrm{M}_{13}\lambda_1 + \mathrm{M}_{23}\lambda_2 = \mathrm{M}_{14}\lambda_1 + \mathrm{M}_{34}\lambda_3 = 0 \ldots\ldots\ldots (\Omega.)$$

For the plane triangle BCD, the equation for the Brocard-circle referred to

the triangle itself may be written (see correction, xx. *Quart. Journal*, p. 59),
$$\frac{(bd)^2}{p_2 p_4}\lambda_2\lambda_4+\frac{(cd)^2}{p_3 p_4}\lambda_3\lambda_4+\frac{(bc)^2}{p_2 p_3}\lambda_2\lambda_3$$
$$-\frac{1}{k}\left\{\frac{(cd)^2(cb)^2}{p_3}\lambda_3+\frac{(cb)^2(bd)^2}{p_2}\lambda_2+\frac{(bd)^2(cd)^2}{p_4}\lambda_4\right\}\left(\frac{\lambda_2}{p_2}+\frac{\lambda_3}{p_3}+\frac{\lambda_4}{p_4}\right)=0,$$
where $k=(cd)^2+(bd)^2+(cb)^2$ and p_2, p_3, p_4 are the respective perpendiculars from the vertices to the opposite sides denoted by (cd), (bd), (bc).

Now, since $\frac{\lambda_2}{p_2}$, $\frac{\lambda_3}{p_3}$, $\frac{\lambda_4}{p_4}$ have the same values, whether we consider the letters as denoting the coordinates and perpendiculars in the plane, or the corresponding tetrahedral coordinates and perpendiculars to opposite faces, the system of spheres through the Brocard-circle is given by
$$\Sigma\frac{(bd)^2}{p_2 p_4}\lambda_2\lambda_4-\frac{1}{k}\left\{\frac{(cd)^2(cb)^2}{p_3}\lambda_3+\frac{(cb)^2(bd)^2}{p_2}\lambda_2+\frac{(bd)^2(cd)^2}{p_4}\lambda_4+\mathrm{T}\frac{\lambda_1}{p_1}\right\}$$
$$\times\left(\frac{\lambda_1}{p_1}+\frac{\lambda_2}{p_2}+\frac{\lambda_3}{p_3}+\frac{\lambda_4}{p_4}\right)=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\omega).$$
In order that the point (Ω) may satisfy (ω), we must have
$$\mathrm{T}=(cb)^2(ad)^2+(bd)^2(ac)^2+(cd)^2(ab)^2,$$
and from symmetry it appears that, if Ω is on the sphere, then the corresponding point before referred to is also on the sphere.

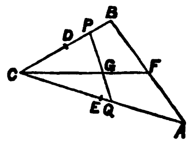
The condition is found by (Ω). For we must have
$$\left\{\frac{\mathrm{M}_{14}}{\mathrm{M}_{34}}\frac{(cd)^2(cb)^2}{p_3}+\frac{\mathrm{M}_{13}}{\mathrm{M}_{23}}\frac{(cb)^2(bd)^2}{p_2}+\frac{\mathrm{M}_{12}}{\mathrm{M}_{24}}\frac{(bd)^2(ad)^2}{p_4}\right\}+\mathrm{T}\frac{\mathrm{T}}{p_1}=0,$$
and $$-\frac{\mathrm{M}_{14}}{\mathrm{M}_{34}}\frac{(cd)^2(cb)^2}{p_3}=-\frac{(ad)^2}{p_1 p_4}\cdot\frac{p_3 p_4}{(cd)^2}\frac{(cd)^2(cb)^2}{p_3}=-\frac{(cb)^2(ad)^2}{p_1},\ \&c.$$
(SALMON's *Geometry of Three Dimensions*, 3rd ed. p. 179).

The three spheres intersect in a Brocard-point of the triangle BCD, because such intersection is that of three circles in its plane such that one touches BD at B and passes through C, and another touches BC at C and passes through D.

The plane of section of the circumscribing sphere and the sphere through the Brocard-circle of the triangle BCD, and (Ω) and its corresponding point, passes through the line $\frac{\lambda_2}{(cd)}+\frac{\lambda_3}{(bd)}+\frac{\lambda_4}{(bc)}=0$ referred to the triangle. This is the line of intersections of the tangents to the circumscribing circle at the vertices with the opposite sides; and the lines joining the feet of the Symmedians also pass respectively through the same intersections.

8119. (By Professor CROFTON, F.R.S.)—If three points are taken at random within a triangle, prove that the chance that the centroid of the triangle lies inside the triangle formed by the three points is $\frac{2}{27}+\frac{20}{81}\log 2$.

Solution by G. Heppel, M.A.



If any area be divided into two parts in the ratio $k : l$; the chance that two random points are on opposite sides of the dividing line is $2kl \div (k+l)^2$.

Let D, E, F be the feet of the medians, G the centroid. Now, chance that the three points are not together in one of the divisions made by CF is $\frac{3}{4}$. Suppose CBF to be the division containing only one point, and through it draw PGQ. The chances of P being in CD, DB, BF are each $\frac{1}{3}$, since the triangles CGD, BGD, BGF are equal. It may be shown by elementary geometry that, in the cases where P divides CD, BD, BF, in each instance in the ratio $m : 1$, the corresponding divisions of CFA are in the ratios $\dfrac{m}{12-10m}$, $\dfrac{4-2m}{8-7m}$, $\dfrac{4-4m}{8-5m}$, respectively. Therefore favourable chance is

$$2\int_0^1 \frac{m\,(12-10m)+(4-2m)\,(8-7m)+(4-4m)\,(8-5m)}{(12-9m)^2}\,dm$$

$$= \tfrac{2}{9}\int_0^1 \frac{64-84m+24m^2}{(4-3m)^2}\,dm = \tfrac{2}{9}\left(\frac{12+20\log 4}{9}\right);$$

therefore, multiplying by $\frac{1}{3}$ and $\frac{3}{4}$, as provided for above, we find the required chance to be $\frac{2}{27}+\frac{20}{81}\log 2$.

8148 & 7754. (By Âsûtosh Mukhopâdhyây, B.A., F.R.A.S.)—(8148.) If a, b, c, d be the sides, δ_1, δ_2 the diagonals, and ϕ the angle of intersection of the diagonals of a spherical quadrilateral, show (1) that

$$\cos\phi = (\cos a\cos c - \cos b\cos d)\,/\,\sin\delta_1\sin\delta_2.$$

(7754.) Herefrom deduce the corresponding theorem in plane trigonometry.

Solution by G. G. Morrice, B.A.; A. Gordon; *and others.*

(8148.) Let A, B, C, D, O be the points of intersection of the sides and diagonals respectively; then we have

$$\begin{aligned}\sin\delta_1\sin\delta_2\cos\phi &= \cos\phi\sin(\mathrm{AO}+\mathrm{OC})\sin(\mathrm{BO}+\mathrm{OD})\\ &= (\sin\mathrm{OA}\sin\mathrm{OB}\cos\mathrm{OC}\cos\mathrm{OD}\\ &\quad+\sin\mathrm{OB}\sin\mathrm{OC}\cos\mathrm{OA}\cos\mathrm{OD}+\sin\mathrm{OA}\sin\mathrm{OD}\cos\mathrm{OC}\cos\mathrm{OB}\\ &\quad+\sin\mathrm{OC}\sin\mathrm{OD}\cos\mathrm{OA}\cos\mathrm{OB})\cos\phi;\end{aligned}$$

$$\begin{aligned}\cos\phi &= (\cos\mathrm{AB}-\cos\mathrm{OA}\cos\mathrm{OB})\,/\,\sin\mathrm{OA}\sin\mathrm{OB}\\ &= (\cos\mathrm{DC}-\cos\mathrm{OC}\cos\mathrm{OD})\,/\,\sin\mathrm{OC}\sin\mathrm{OD}\\ &= (-\cos\mathrm{AD}+\cos\mathrm{OA}\cos\mathrm{OD})\,/\,\sin\mathrm{OA}\sin\mathrm{OD}\\ &= (-\cos\mathrm{BC}+\cos\mathrm{OB}\cos\mathrm{OC})\,/\,\sin\mathrm{OB}\sin\mathrm{OC}.\end{aligned}$$

Let arc $AB = a$, $BC = b$, &c.

therefore $(\cos a - \cos OA \cos OB)(\cos c - \cos OC \cos OD)$

$$= (\cos b - \cos OB \cos OC)(\cos d - \cos OA \cos OD).$$

Substituting in this for $\cos a$, &c., we obtain the above expression for $\sin\delta_1 \sin\delta_2 \cos\phi$.

(7764.) For the theorem in plane trigonometry, write, for $\cos a$, $1-\frac{1}{2}a^2$, &c.; for $\sin\delta_1$, δ_1, &c.; then $4\cos\phi\,.\,\delta_1\delta_2 = a^2c^2 - b^2d^2 - 2(a^2+c^2) + 2(b^2+d^2)$.

7902. (By R. Rawson, F.R.A.S., &c.)—Show that the general integral of $(A_2)^{\frac{1}{2}}\left(1+B_2x^2+\dfrac{B_2{}^2A_4}{A_2{}^2}x^4\right)^{\frac{1}{2}}\cdot\dfrac{dy}{dx}+(B_2)^{\frac{1}{2}}(1+A_2y^2+A_4y^4)^{\frac{1}{2}}=0$

is $[CA_2+B_2A_4x^2]\,y^2+2\,[(A_2B_2)^{\frac{1}{2}}(A_4+C^2-CA_2)^{\frac{1}{2}}]\,xy+A_2+CB_2x^2=0.$

Solution (by verification) by the Proposer.

The general integral admits of the form

$$u = \left\{CA_2+B_2A_4x^2\right\}y^2+2\left\{(A_2B_2)^{\frac{1}{2}}(A_4+C^2-CA_2)^{\frac{1}{2}}\right\}xy+A_2+CB_2x^2=0 \quad\text{.............(1)},$$

$$= \left\{CB_2+B_2A_4y^2\right\}x^2+2\left\{(A_2B_2)^{\frac{1}{2}}(A_4+C^2-CA_2)^{\frac{1}{2}}\right\}yx+A_2+CA_2y^2=0 \quad\text{.............(2)}.$$

But
$$\frac{du}{dy}\frac{dy}{dx}+\frac{du}{dx}=0 \quad\text{..............................(3)}.$$

Differentiating (1) and (2), we have

$$\frac{du}{dy} = 2\,(CA_2+B_2A_4x^2)\,y+2\left\{(A_2B_2)^{\frac{1}{2}}(A_4+C^2-CA_2)^{\frac{1}{2}}\right\}x,$$

$$\frac{du}{dx} = 2\,(CB_2+B_2A_4y^2)\,x+2\left\{(A_2B_2)^{\frac{1}{2}}(A_4+C^2-CA_2)^{\frac{1}{2}}\right\}y.$$

From (1) and (2), by the solution of a quadratic,

$$2\,(CA_2+B_2A_4x^2)\,y+2\left\{(A_2B_2)^{\frac{1}{2}}(A_4+C^2-CA_2)^{\frac{1}{2}}\right\}x = 2A_2(-C)^{\frac{1}{2}}\left(1+B_2x^2+\frac{B_2{}^2A_4}{A_2{}^2}x^4\right)^{\frac{1}{2}},$$

$$2\,(CB_2+B_2A_4y^2)\,x+2\left\{(A_2B_2)^{\frac{1}{2}}(A_4+C^2-CA_2)^{\frac{1}{2}}\right\}y = 2\,(A_2B_2)^{\frac{1}{2}}(-C)^{\frac{1}{2}}(1+A_2y^2+A_4y^4)^{\frac{1}{2}}.$$

Substitute these values in (3), then the verification of the property in the question is manifest.

If $A_2 = B_2 = -1-K^2$, and $A_4 = K^2$,

$$(C+K^2x^2)\,y^2+2\,(1+C)^{\frac{1}{2}}\,(C+K^2)^{\frac{1}{2}}\,xy+1+Cx^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

is the general solution of the standard form

$$(1-x^2\,.\,1-K^2x^2)^{\frac{1}{2}}\cdot\frac{dy}{dx}+(1-y^2\,.\,1-K^2y^2)^{\frac{1}{2}} = 0 \quad\ldots\ldots\ldots\ldots\ldots(5).$$

Other interesting theorems readily follow from the property in the question.

7472. (By H. G. Dawson, B.A.)—Denoting by α_1, α_2, α_3, α_4, α_5, the the roots of $x^5+p_1x^4+p_2x^3+p_3x^2+p_4x+p_5 = 0$; show that the equation whose roots are the 10 products α_1, α_2, &c., may be obtained by substituting for a, b c, f, g, h, respectively, in the equation $abc+2fgh-af^2-bg^2-ch^2=0$, the quantities,

$$2,\ 2\left\{p_2+\frac{p_1p_5}{y^2}-y\left(1+\frac{p_4}{y^2}\right)\right\},\ 2\left\{p_4+\frac{p_3p_5}{y^2}-\frac{p_5}{y}\left(p_1+\frac{p_5}{y^2}\right)\right\},$$

$$p_3+\frac{p_2p_5}{y^2}-2\,\frac{p_5}{y},\ y\left(1+\frac{p_4}{y}\right),\ p_1+\frac{p_5}{y^2}.$$

Solution by the Proposer.

It will be necessary first to show that the roots α, β, γ, δ, ϵ, ζ, of the equation $x^6+q_1x^5+q_2x^4+q_3x^3+q_4x^2+q_5x+q_6 = 0$, satisfy the condition

$$\alpha\beta\gamma = \delta\epsilon\zeta \text{ if } \begin{vmatrix} 2, & q_1, & \dfrac{q_5}{\sqrt{q_6}} \\ q_1, & 2\left(q_2-\dfrac{q_5}{\sqrt{q_6}}\right), & q_3-2\sqrt{q_6} \\ \dfrac{q_5}{\sqrt{q_6}}, & q_3-2\sqrt{q_6}, & 2\,(q_4-q_1\sqrt{q_6}) \end{vmatrix} = 0;$$

for, putting the sextic $\equiv (x^3+ax^2+bx+c)(x^3+a'x^2+b'x+c')$, multiplying and identifying, we get after a few reductions

$$a+a' = q_1, \qquad b+b' = \frac{q_5}{\sqrt{q_6}},$$

$$ab'+ba' = q_3-2\sqrt{q_6}, \quad bb' = q\ -q_1\sqrt{q_6}, \quad aa' = q_2-\frac{q_5}{\sqrt{q_6}}.$$

But $\begin{vmatrix} 1, & 1, & 0 \\ a, & a', & 0 \\ b, & b', & 0 \end{vmatrix} \times \begin{vmatrix} 1, & 1, & 0 \\ a', & a, & 0 \\ b', & b, & 0 \end{vmatrix} \equiv 0, \therefore \begin{vmatrix} 2, & a+a', & b+b' \\ a+a', & 2aa', & ab'+a'b \\ +b', & ab'+b'a, & 2bb' \end{vmatrix} \equiv 0,$

whence above condition follows at once.

Let the quintic in question be multiplied by $x+\dfrac{p_5}{y^2}$; suppose resulting

sextic to have its roots connected by above relation, then

$$\alpha_1\alpha_2\alpha_3 = \alpha_4\alpha_5\left(-\frac{p_5}{y^2}\right), \quad \therefore \quad y = \pm\,\alpha_4\alpha_5,$$

and, applying the determinant condition to the coefficients of the sextic, we get

$$\begin{vmatrix} 2, & p_1+\dfrac{p_5}{y^2}, & y\left(1+\dfrac{p_4}{y^2}\right)i \\ p_1+\dfrac{p_5}{y^2}, & 2\left\{p_2+\dfrac{p_1p_5}{y^2}-y\left(1+\dfrac{p_4}{y^2}\right)\right\}, & p_3+\dfrac{p_2p_5}{y^2}-2i\dfrac{p_5}{y} \\ y\left(1+\dfrac{p_4}{y^2}\right)i, & p_3+\dfrac{p_2p_5}{y^2}-2i\dfrac{p_5}{y}, & 2\left\{p_4+\dfrac{p_3p_5}{y^2}-i\dfrac{p_5}{y}\left(p_1+\dfrac{p_5}{y^2}\right)\right\} \end{vmatrix}$$

$$= 0, \text{ where } i^2 = 1.$$

If $i = +1$, we get the equation whose roots are $+\alpha_1\alpha_2$, &c.; if $i = -1$, we get the equation whose roots are $-\alpha_1\alpha_2$, &c.

Note on the Lister-Process Question. *By* Professor Crofton, F.R.S.

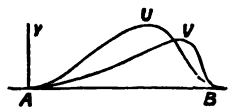
This Question was proposed [see Vol. 37, p. 40] in the following form:—

6929. (By Dr. Donald McAlister.)—"If Lister's process has been found to succeed in m cases and fail in n, and the ordinary dressing to succeed in p cases and fail in q; to find the probability that the former process is really superior."

Of course, if the numbers m, n, p, q are very large, there is no doubt on the matter, as James Bernouilli's theorem shows that the real facility of the event tends to become equal to the number of successes divided by the whole number of trials: so that if $m : n$ is greater than $p : q$, Lister's process gives a greater facility for a cure.

In other cases, if x = unknown facility (or real probability) of a success by Lister's process, the curve of frequency for x is known to be, by Bayes' theorem, $\quad y = x^m(1-x)^n$ ....(1);

or the real facility of a success is equally likely to be the abscissa of any point taken at random within the area of the curve (1). Likewise the facility of a success by the usual process is equally likely to be the abscissa of any point taken within the curve



$$y = x^p(1-x)^q \quad \text{..................................(2).}$$

Let these curves be AVB, AUB, taking AB = 1. Then the question as to the probability that the facility of a cure by Lister's process exceeds the facility in the usual case, is simply "What is the chance that the abscissa of a point taken at random in AVB exceeds that of a point taken in AUB?"

The whole number of cases of two such points is the product of the two areas; *i.e.*,

$$\int_0^1 x^m (1-x)^n dx \times \int_0^1 x'^p (1-x')^q dx' = \frac{m!\, n!\, p!\, q!}{(m+n+1)!\,(p+q+1)!},$$

while the number of favourable cases is the number of pairs of points, one in AVB, one in AUB, in which the abscissa of the former exceeds that of the latter. This will be

$$\int_0^1 x^m (1-x)^n dx \int_0^x x'^p (1-x')^q dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and the required probability is found by dividing this by the whole number of cases above.

The integration, when the numbers are large, may be somewhat troublesome and tedious, but of course is always possible in finite terms.

Suppose Lister's process to have succeeded in three cases out of four; and the other in two cases out of four; the probability that the former is the better (*i.e.*, gives greater facility for a cure) is

$$\varpi = \int_0^1 x^3 (1-x)\, dx \int_0^x x'^2 (1-x')^2 dx' \div \frac{3!\,2!\,2!}{[5!]^2!}.$$

Hence $\varpi = \frac{31}{42}$.

Had it been proposed to find the chance that Lister's process excels the other by at least a given difference; say, that it cures at least $\frac{1}{10}$ more patients out of the whole (large) number operated on; the problem is unchanged, except that the upper limit of the second integral (3) is $x - \frac{1}{10}$ instead of x.

4493. (By Professor Martin, M.A., Ph.D.)—A cylinder of radius r *rolls* down the convex surface of a fixed cylinder, radius R; find the point of separation.

Solution by the Proposer.

Let β be the angular distance of the rolling cylinder from the upper extremity of a vertical diameter of the fixed one when motion begins, ϕ its angular distance when it leaves the fixed cylinder, v its velocity at the point of separation, Q the reaction of the curve at any point, x and y the vertical and horizontal axes of the parabola the axis of the rolling cylinder moves in after leaving the fixed one, and ρ the radius of curvature of any point of the parabola.

Then $Q = \dfrac{v^2}{\rho} - g\dfrac{dy}{ds}$, $\dfrac{dy}{ds} = \cos\phi$, and at the point of separation $Q = 0$,

and $\rho = R + r$; therefore $v^2 = g(R+r)\cos\phi$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$,

$$(R+r)(\cos\rho - \cos\phi) = \text{vertical descent of the rolling cylinder} = h.$$

The velocity of the cylinder is the same as it would be if it had rolled down an inclined plane, height h.

Let w = distance the cyclinder would *roll* down a plane of inclination λ in the time t, θ = the whole angle through which the cyclinder would revolve about its centre of gravity, k = its radius of gyration about its axis, F = the friction of the plane on the cylinder, and m = the mass of the cylinder. Then, for the motion of the cylinder, we have

$$\frac{md^2w}{dt^2} = mg\sin\lambda - \mathrm{F} \quad\text{.........................(2)},$$

and, taking moments about the centre of gravity of the cylinder,

$$\frac{mk^2d^2\theta}{dt^2} = \mathrm{F}r^2 \quad\text{.................................(3)}.$$

But, since F is sufficiently great to secure perfect rolling, we must evidently have $w = r\theta$; and therefore, by (3), $\dfrac{mk^2d^2w}{dt^2} = \mathrm{F}r^2$; and thence,

from (2), we get $$(r^2+k^2)\frac{d^2w}{dt^2} = r^2g\sin\lambda \quad\text{...........................(4)}.$$

Multiplying by $2w$ and integrating,

$$\left(\frac{dw}{dt}\right)^2 = \frac{2r^2gw\sin\lambda}{r^2+k^2} = v^2 \quad\text{...........................(5)}.$$

But $k^2 = \frac{1}{2}r^2$, therefore $v = (\frac{4}{3}gw\sin\lambda)^{\frac{1}{2}} = (\frac{4}{3}gh)^{\frac{1}{2}}$, since $w\sin\lambda$ = vertical descent of the cylinder.

Equating the values of v^2 given by (1) and (5),

$g(\mathrm{R}+r)\cos\phi = \frac{4}{3}gh = \frac{4}{3}g(\mathrm{R}+r)(\cos\beta-\cos\phi)$; whence $\cos\phi = \frac{4}{7}\cos\beta$.

Cor.—When the rolling cylinder starts from the highest point of the surface of the fixed one, $\beta=0$, $\cos\beta = 1$, and $\cos\phi = \frac{4}{7}$.

8209. (By B. Hanumanta Rau, B.A.)—If straight lines AD, BE, CF are drawn at right angles to the sides CA, AB, BC of a triangle, and R′, R denote the circum-radii of the triangles DEF, ABC; prove (1) that $\mathrm{R}' = \mathrm{R}\cot\theta$, where θ is the Brocard angle of the triangle ABC; and (2) express the ratio of ΔDEF, ΔABC in terms of the same angle.

Solution by Capt. H. Brocard.

Plus généralement, si les lignes AD, BE, CF font les mêmes angles ϕ avec les côtés en suivant le périmètre dans un sens déterminé, le triangle DEF est semblable au proposé, et le rapport de similitude est

$$\frac{\mathrm{EF}}{\mathrm{CB}} = \frac{\mathrm{EC}-\mathrm{CF}}{\mathrm{CB}} = \frac{\sin(\mathrm{B}-\phi)}{\sin\mathrm{B}} - \frac{b\sin\phi}{a\sin\mathrm{C}}$$

$$= \cos\phi - \frac{\sin^2\mathrm{B}+\sin\mathrm{A}\sin\mathrm{C}\cos\mathrm{B}}{\sin\mathrm{A}\sin\mathrm{B}\sin\mathrm{C}}\sin\phi = \cos\phi - \cot\theta\sin\phi = \frac{\sin(\theta-\phi)}{\sin\theta},$$

θ étant l'angle dont la cotangente est

$$\cot A + \cot B + \cot C = \frac{1 + \cos A \cos B \cos C}{\sin A \sin B \sin C}.$$

Ce rapport devient $\cot \theta$ dans le cas où ϕ est pris égal à 90°, ce qui est la condition de l'énoncé. On en déduit une construction de l'angle ϕ qui n'est autre que celle que j'ai donnée pour la première fois en 1877 dans la *Nouvelle Corresp. Math.*, T. III., p. 187.

Le triangle DEF se réduit à l'un des points ω, ω', lorsque $\phi = \theta$.

Quant à l'expression de $\cot \theta$ en fonction des côtés du triangle, on n'a, pour l'obtenir, qu'à remplacer $\cos A$, $\sin A$, ... etc., par leurs valeurs $\frac{b^2+c^2-a^2}{2bc}$, $\frac{a}{D}$, etc., D étant le diamètre du cercle ABC. L'on trouve ainsi $\cot \theta = D \frac{a^2+b^2+c^2}{2abc}$.

8196. (By Arthur Hill Curtis, LL.D., D.Sc. Suggested by Question 8164.)—Prove that $\int_0^k \frac{\sin \alpha \, dx}{1 + 2x \cos \alpha + x^2} = \lambda (\alpha - 2n\pi)$, where n is as in Question 8164, and λ is determined by the equation

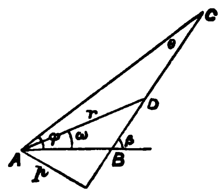
$$\frac{\sin \lambda (\alpha - 2n\pi)}{\sin (1-\lambda)(\alpha - 2n\pi)} = k.$$

8164. (By Professor Ch. Hermite, LL.D.)—Démontrer

$$\int_0^1 \frac{\sin \alpha \, dx}{1 + 2x \cos \alpha + x^2} = \tfrac{1}{2}\alpha - n\pi.$$

On désigne par n le plus grand nombre entier par excès ou par défaut dans $\frac{\alpha}{2\pi}$.

Solution by Arthur Hill Curtis, LL.D., D.Sc.



Let $\alpha = 2n\pi \pm \beta$; construct the triangle ABC, whose vertical angle $B = \pi - \beta$, side $AB = 1$, and side $BC = k$. Denoting by ϕ the angle BAC, by θ the angle ACB, by r the radius vector from A to any point D taken on BC, by ω the angle DAB, and by p the perpendicular from A on BC, we have

$$\int_0^k \frac{\sin \alpha \, dx}{1 + 2x \cos \alpha + x^2} = \pm \int_0^k \frac{\sin \beta \, dx}{1 + 2x \cos \beta + x^2}$$

$$= \pm \int_0^k \frac{p \, dx}{r^2} = \pm \int_0^k \frac{r^2 \, d\omega}{r^2} = \pm \int_0^\phi d\omega = \pm \phi;$$

let $\phi = \lambda\beta$, therefore $\pm\phi = \pm \lambda\beta = \lambda(\alpha - 2n\pi)$, where λ is determined from the relation

$$\frac{\sin \lambda\beta}{\sin (1-\lambda)\beta} = \frac{\sin \phi}{\sin (\beta - \phi)} = \frac{\sin \phi}{\sin \theta} = k, \quad \text{or} \quad \frac{\sin \lambda (\alpha - 2n\pi)}{\sin (1-\lambda)(\alpha - 2n\pi)} = k.$$

7772 & 7840. (By Professor Wolstenholme, M.A., Sc.D.)—7772. The curve whose equation is $x^5+y^5+3ax^2y^2=a^3xy$, has a loop of area $\frac{3}{10}a^2$; and this is also the area between the curve and its asymptote $5x+5y+3a=0$. If tangents AA′C, BB′C be drawn parallel to the axes Oy, Ox, touching the loop in A′, B′, the area between the curve AO and AA′ is $\frac{1}{50}a^2$; between the curve and AO′ is $\frac{3}{50}a^2$; between OA′, OB and the curve is $\frac{9}{50}a^2$; and between the curve and A′C, B′C is

$$\tfrac{1}{50}a^2\left(2^{\frac{21}{5}}-17\right)=\cdot 0275834a^2.$$

7840. Prove that (1) the curve whose equation is $x^{2n+1}+y^{2n+1}\ldots ax^ny^n$, when n is positive, consists of a loop and (generally) an infinite branch; (2) the area of the loop, and also the area included between the infinite branch and the asymptote $x+y=\dfrac{a(-1)^n}{2n+1}$, is $\dfrac{a^2}{2(2n+1)}$; (3) if O be the origin, AA′C, BB′C tangents parallel to the coordinate axes touching the curve in A′, B′ and meeting the coordinate axes in A, B, the area AOA′ (outside the curve) is $\dfrac{a^2}{2}\,\dfrac{n^2}{(2n+1)^2}$, the area cut off by OA′ is $\dfrac{a^2}{2}\,\dfrac{n}{(2n+1)^2}$, and this is also the area included between one of the axes, the asymptote, and the infinite branch; and (4) the area A′CB′ (outside the loop) and the coordinates of the centroid are respectively

$$a^2\left\{\frac{n(n+1)}{(2n+1)^2}\left[\left(\frac{n+1}{n}\right)^{\frac{1}{2n+1}}-1\right]-\frac{1}{2(2n+1)^2}\right\},$$

$$x=y=\frac{a}{3}\,\frac{n(n+1)}{(2n+1)^2}\,\frac{\pi}{\sin\dfrac{n\pi}{2n+1}}.$$

Solution by D. Edwardes; W. T. Mitchell, M.A.; *and others.*

7772. The equation is satisfied by $x=\dfrac{a\lambda^4}{1+\lambda^5}$, $y=\dfrac{a\lambda}{1+\lambda^5}$. Then

$$d\,\frac{x}{y}=3\lambda^2d\lambda=\frac{y\,dx-x\,dy}{y^2},\ \text{so that}\ r^2d\theta=-3\lambda^2y^2d\lambda=-\frac{3a^2\lambda^4d\lambda}{1+\lambda^5}.$$

Therefore $$\int r^2\,d\theta=\frac{3a^2}{5(1+\lambda^5)}+\text{constant}.$$

If $\dfrac{y}{x}=\tan\theta$, when $\theta=0$, $\lambda=\infty$, and when $\theta=\frac{1}{2}\pi$, $\lambda=0$. Hence, area of loop $=\frac{3}{10}a^2\left[\dfrac{1}{1+\lambda^5}\right]_\infty^0=\frac{3}{10}a^2$. Again, $\dfrac{dy}{dx}=\dfrac{1-4\lambda^5}{\lambda^3(4-\lambda^5)}$. Hence, for A′, $\lambda^5=4$, and we easily find that OA = OB. Hence, area between OA′ and curve $=\frac{3}{10}a^2\left[\dfrac{1}{1+\lambda^5}\right]_{\lambda=\infty}^{\lambda^5=4}=\frac{3}{50}a^2$.

We have $$OA=\frac{4^{\frac{4}{5}}}{5}a,\quad AA'=\frac{4^{\frac{1}{5}}}{5}a.$$

Whence area of triangle OAA′ $=\frac{2}{25}a^2$. Hence, area between OA, AA′,

and curve $= \dfrac{a^2}{50}$. Area between OA′, OB′, and curve

$$= \tfrac{9}{10}a^2 \left[\frac{1}{1+\lambda^5}\right]_{\lambda^5=4}^{\lambda^5=\frac{1}{4}} = \tfrac{9}{50}a^2.$$

Area of quadrilateral OA′CB′ is easily found to be $\frac{1}{25}a^2\,(4^{\frac{3}{5}}-4)$. Hence, area between CA′, CB′, and curve $= \frac{1}{50}a^2\,(2^{\frac{11}{5}}-17)$. When the axes are turned through 45°, the equations of curve are

$$x\sqrt{2} = \frac{a\lambda\,(1+\lambda^3)}{1+\lambda^5}, \quad y\sqrt{2} = \frac{a\lambda\,(1-\lambda^3)}{1+\lambda^5}.$$

Also the length of the perpendicular from the origin upon the asymptote is $\dfrac{3a}{5\sqrt{2}}$. Then, area between curve and asymptote

$$= 2\int_0^{-\frac{3a}{5\sqrt{2}}} y\,dx = a^2\int_0^{-1} \frac{\lambda\,(1-\lambda^3)}{1+\lambda^5}\,d\left\{\frac{\lambda\,(1+\lambda^3)}{1+\lambda^5}\right\}$$

$$= a^2\,\frac{5\lambda^2\,(1-\lambda^6)-6\,(1+\lambda^5)}{10\,(1+\lambda^5)^2},$$

or, dividing out the factor $(1+\lambda)^2$,

$$= a^2\,\frac{-5\lambda^6+10\lambda^5-15\lambda^4+14\lambda^3-13\lambda^2+12\lambda-6}{10\,(\lambda^4-\lambda^3+\lambda^2-\lambda+1)^2},$$

and this, taken between the limits -1 and 0, gives $\frac{3}{10}a^2$, the same as the area of the loop.

7840. The equation is satisfied by

$$x = a\,\frac{\lambda^{n+1}}{1+\lambda^{2n+1}}, \quad y = a\,\frac{\lambda^n}{1+\lambda^{2n+1}},$$

and $$\frac{dy}{dx} = \frac{n-(n+1)\,\lambda^{2n+1}}{(n+1)\,\lambda-n\lambda^{2n+2}}, \quad \text{OA} = a\,\frac{n}{2n+1}\left(\frac{n+1}{n}\right)^{\frac{n+1}{2n+1}},$$

$$\text{AA}' = a\,\frac{n}{2n+1}\left(\frac{n+1}{n}\right)^{\frac{n}{2n+1}},$$

and similarly for OB, BB′. OBCA is not in this case a square. The work is merely a repetition of the above. The area between curve and asymptote is

$$2\int_0^{-\frac{a}{\sqrt{2}\,(2n+1)}} y\,dx = a^2\int_0^{-1} \frac{\lambda^n-\lambda^{n+1}}{1+\lambda^{2n+1}}\,d\left(\frac{\lambda^n(1+\lambda)}{1+\lambda^{2n+1}}\right)$$

$$= a^2\,\frac{(2n+1)\,(\lambda^{2n}-\lambda^{2n+2})-2-2\lambda^{2n+1}}{2\,(2n+1)\,(1+\lambda^{2n+1})^2}.$$

The limit of this, when $\lambda = -1$, is $-\dfrac{a^2}{2\,(2n+1)}$, and therefore, &c.

Let (x), (y) be the coordinates of the centroid, and A denote the area

of the loop. Then $A(x) = \int \tfrac{1}{3} r \,.\, r^2\, d\theta$ taken over the area

$$= \tfrac{1}{3}a^3 \int_0^\infty \frac{\lambda^{3n+1}}{(1+\lambda^{2n+1})^3}\, d\lambda$$

$$= -\frac{a^3}{6(2n+1)} \left\{ \left[\frac{\lambda^{n+1}}{(1+\lambda^{2n+1})^2}\right]_0^\infty - (n+1)\int_0^\infty \frac{\lambda^n d\lambda}{(1+\lambda^{2n+1})^2} \right\}$$

$$= \frac{a^3(n+1)}{6(2n+1)} \int_0^\infty \frac{\lambda'^{3n} d\lambda'}{(1+\lambda'^{2n+1})^2} \left(\text{substituting } \frac{1}{\lambda'} \text{ for } \lambda\right)$$

$$= -\frac{a^3(n+1)}{6(2n+1)^2} \left\{ \left[\frac{\lambda'^n}{1+\lambda'^{2n+1}}\right]_0 - n\int_0^\infty \frac{\lambda'^{n-1} d\lambda'}{1+\lambda'^{2n+1}} \right\}$$

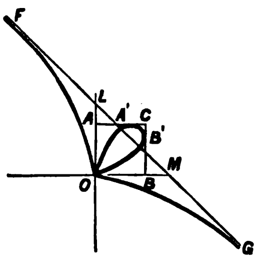
$$= \frac{a^3 n(n+1)}{6(2n+1)^2} \int_0^\infty \frac{\lambda'^{n-1} d\lambda'}{1+\lambda'^{2n+1}} = \frac{a^3 \,.\, n(n+1)}{6(2n+1)^3} \cdot \frac{\pi}{\sin\dfrac{n\pi}{2n+1}}.$$

Therefore (x) = result stated. Similarly for (y), observing that

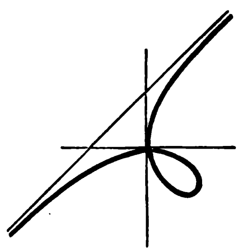
$$\sin\frac{n+1}{2n+1}\pi = \sin\frac{n\pi}{2n+1}.$$

[From the equation $x^{2n+1} + y^{2n+1} = ax^n y^n$, we see that the origin (O) is a multiple point, and that it is clearly the only one; also the tangents at O are the axes; in the first quadrant, the radius-vector (r) is positive, finite, and continuous, and it is zero when $\theta = 0$ or $\tfrac{1}{2}\pi$, and hence r has a maximum value somewhere in this quadrant, which therefore contains a loop of the curve. Since, in general, x and y cannot both be negative, the curve has no branch in the third quadrant; and, since the equation $1 + \mu^{2n+1} = 0$ gives the inclination to the axis of x of all the asymptotes, there is but one whose equation can readily be found to be

$$x + y = a\frac{(-1)^n}{2n+1}.$$



(Fig. 1.)


(Fig. 2.)

The curve evidently touches this asymptote at infinity in the second quadrant, and also in the fourth; it is therefore of the form in Fig. 1. An alternative form, n still positive, is shown in Fig. 2.]

4457. (By Walter Siverly.)—The first of two casks contains a gallons of wine, and the second b gallons of water; c gallons are drawn from the second cask, and then c gallons are drawn from the first cask and poured into the second, and the deficiency in the first supplied with c gallons of water; c gallons are then drawn from the first cask, and c gallons drawn from the second and poured into the first and the deficiency in the second supplied with c gallons of water. Required the quantity of wine in each cask after n such operations as described above.

Solution by Prof. Martin, M.A., Ph.D.; *the* Proposer; *and others.*

Let U_n, V_n be the wine in the first and second casks respectively after n operations, then we have

$$\left(1-\frac{c}{a}\right)^2 U_n + \frac{c}{b}\left(1-\frac{c}{b}\right) V_n + \frac{c^2}{ab} U_n = U_{n+1} \ldots\ldots\ldots\ldots (1),$$

$$\left(1-\frac{c}{b}\right)^2 V_n + \frac{c}{a}\left(1-\frac{c}{b}\right) U_n = V_{n+1} \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

whence
$$\left(1-\frac{c}{b}\right)^2 V_{n+1} + \frac{c}{a}\left(1-\frac{c}{b}\right) U_{n+1} = V_{n+2} \ldots\ldots\ldots\ldots\ldots (3).$$

From (2) and (3), we get

$$U_n = \frac{V_{n+1}}{\frac{c}{a}\left(1-\frac{c}{b}\right)} - \frac{\left(1-\frac{c}{b}\right)V_n}{\frac{c}{a}}, \quad U_{n+1} = \frac{V_{n+2}}{\frac{c}{a}\left(1-\frac{c}{b}\right)} - \frac{\left(1-\frac{c}{b}\right)V_{n+1}}{\frac{c}{a}}.$$

Substituting these values of U_n and U_{n+1} in (1), there results

$$V_{n+2} - \left\{\left(1-\frac{c}{b}\right)^2 + \left(1-\frac{c}{a}\right)^2 + \frac{c^2}{ab}\right\} V_{n+1} + \left(1-\frac{c}{a}\right)^2\left(1-\frac{c}{b}\right)^2 = 0.$$

Put the coefficient of $V_{n+1} = A$, and of $V_n = b$; then, by integration,

$$V_n = C \cdot \tfrac{1}{2}\left[A + (A^2-4B)^{\frac{1}{2}}\right]^n + C_1 \cdot \tfrac{1}{2}\left[A - (A^2-4B)^{\frac{1}{2}}\right]^n.$$

When $n = 0$, $V_n = 0$; therefore $C + C_1 = 0$; and when $n = 1$,

$$V_n = C\left(1-\frac{c}{b}\right);$$

therefore
$$C\left(\frac{A+[A^2-4B]^{\frac{1}{2}}}{2}\right) + C_1\left(\frac{A-[A^2-4B]^{\frac{1}{2}}}{2}\right) = C\left(1-\frac{c}{b}\right);$$

whence
$$C = \frac{C\left(1-\frac{c}{b}\right)}{[A^2-4B]^{\frac{1}{2}}}, \quad C_1 = -\frac{C\left(1-\frac{c}{b}\right)}{[A^2-4B]^{\frac{1}{2}}},$$

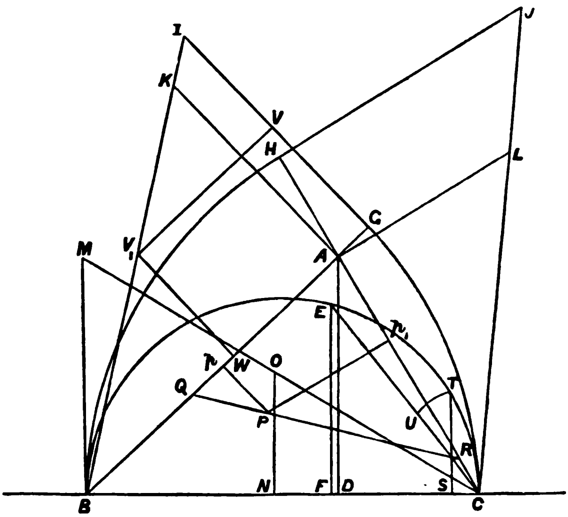
$$V_n = \frac{C\left(1-\frac{c}{b}\right)}{[A^2-4B]^{\frac{1}{2}}}\left\{\left(\frac{A+[A^2+4B]^{\frac{1}{2}}}{2}\right)^n - \left(\frac{A-[A^2-4B]^{\frac{1}{2}}}{2}\right)^n\right\};$$

also,
$$U_n = \frac{2a\left(1-\frac{c}{a}\right)^2 + \frac{2c^2}{b} - aA[A^2-4B]^{\frac{1}{2}}}{2[A^2-4B]^{\frac{1}{2}}}\left\{\left(\frac{A+[A^2-4B]^{\frac{1}{2}}}{2}\right)^n\right\}$$

$$+ \frac{a[A^2-4B]^{\frac{1}{2}} + aA - 2a\left(1-\frac{c}{a}\right)^2 - \frac{2c^2}{b}}{2[A^2-4B]^{\frac{1}{2}}}\left\{\left(\frac{A-[A^2-4B]^{\frac{1}{2}}}{2}\right)^n\right\}.$$

8272. (By Emily Perrin.)—**Given a triangle ABC, and any point P in its plane, give a geometrical construction to determine the line through P which shall bisect the area of the triangle.**

Solution by D. Biddle.



Let ABC be the triangle, and P the point, of which Pp and Pp_1 are coordinates. On BC describe the semicircle BETC, and make CE = AD (the perpendicular from A on BC); also draw the perpendicular EF. Produce BA to G, and CA to H, making BG = CH = BC. Draw GI, HJ at right angles to BG, CH, making $GI = 4Pp$, and $HJ = 2Pp_1$; join BI, CJ, and draw AK, AL parallel to GI, HJ. Draw BM at right angles to BC, making BM = AK. Join CM, and make CN = AL. Draw NO parallel to BM, and make FS = NO. Draw ST at right angles to BC, to cut the semicircle in T, and make CU = CT. Also make GV = EU, and draw VV_1, V_1W parallel to BG, GI. Finally, make AQ = BW, and draw QR through P. QR is the line required.

For, BC being unity,

$$CF = CE^2 = AD^2;\quad AK = 4AB\,.\,Pp;\quad AL = 2AC\,.\,Pp_1;$$

$$NO = 8AB\,.\,AC\,.\,Pp\,.\,Pp_1;\quad CS = AD^2 - NO;\quad EU = AD - (AD^2 - NO)^{\frac{1}{2}};$$

and $BW = EU \,/\, 4Pp$; $\therefore\ AQ = [AD - (AD^2 - 8AB\,.\,AC\,Pp\,Pp_1)^{\frac{1}{2}}] \,/\, 4Pp$,

which is the value arrived at by treating algebraically the two facts, that

$$2AQ\,.\,AR = AB\,.\,AC, \quad\text{and}\quad AQ\,.\,Pp + AR\,.\,Pp_1 = \tfrac{1}{2}AD\,.\,BC.$$

7777. (By Professor Cochez.)—Trouver la courbe dont le rayon de courbure est proportionnel à la puissance $p^{\text{ième}}$ de la normale.

Solution by W. T. Mitchell, M.A.; *and* B. Hanumanta Rau, M.A.

Let $\dfrac{dy}{dx} = q$; then radius of curvature $= -\dfrac{dx}{dq}(1+q^2)^{\frac{3}{2}}$, and Normal

$= y(1+q^2)^{\frac{1}{2}}$, $\therefore\ -\dfrac{dx}{dq}(1+q^2)^{\frac{3}{2}} \propto y^p(1+q^2)^{\frac{1}{2}p} = \dfrac{y^p}{a^{p-1}}\cdot(1+q^2)^{\frac{1}{2}p}$ suppose.

But $\dfrac{dx}{dq} = \dfrac{dx}{dy}\cdot\dfrac{dy}{dq} = \dfrac{1}{q}\cdot\dfrac{dy}{dq}$, $\therefore\ -a^{p-1}\dfrac{dy}{y^p} = (1+q^2)^{\frac{1}{2}(p-3)}\cdot q\,dq$.

Integrating, $c + \dfrac{a^{p-1}}{y^{p-1}} = (1+q^2)^{\frac{1}{2}(p-1)}$, $1+q^2 = \dfrac{(cy^{p-1}+a^{p-1})^{\frac{2}{p-1}}}{y^2}$,

$$\frac{dy}{dx} = \frac{1}{y}\left[(cy^{p-1}+a^{p-1})^{\frac{2}{p-1}} - y^2\right]^{\frac{1}{2}},\quad dx = \frac{\frac{1}{2}dy^2}{\left[(cy^{p-1}+a^{p-1})^{\frac{2}{p-1}} - y^2\right]^{\frac{1}{2}}}.$$

If $c = 0$, $dx = \dfrac{dy^2}{(a^2-y^2)^{\frac{1}{2}}}$, $\therefore\ x = -(a^2-y^2)^{\frac{1}{2}}$ or $x^2+y^2 = a^2$ (a circle).

7529. (By D. Edwardes.)—A uniform elliptic disc of small thickness is projected upon a rough horizontal plane, the friction upon an element being proportional to its area, and the cube of its velocity. If u, v be the components along the axes of the velocity of centre at time t, and w the angular velocity, prove (1) the equations (α being a constant)

$$\frac{du}{dt} + \alpha\left[u(u^2+v^2) + \tfrac{1}{4}w^2(a^2+3b^2)\,u\right] = 0,$$

$$\frac{dv}{dt} + \alpha\left[v(u^2+v^2) + \tfrac{1}{4}w^2(3a^2+b^2)\,v\right] = 0;$$

and (2), if the disc be circular, the path of the centre is a straight line.

Solution by the Proposer.

1. Let x, y be the coordinates of a point referred to the centre of the disc, then, if V be the velocity of the element, $V^2 = (v+wx)^2 + (u-wy)^2$. If m be the mass per unit area, and X, Y the forces on the element, then

$$-X = \mu m V^2 (u - wy)\,dx\,dy, \qquad -Y = \mu m V^2 (v + wx)\,dx\,dy.$$

Hence, if $\overline{X}$, $\overline{Y}$ be the resultant forces on the disc in the directions of the axes,

$$-\frac{\overline{X}}{\mu m} = \iint\left[u^2+v^2+2w(vx-uy)+w^2(x^2+y^2)\right](u-wy)\,dx\,dy,$$

$$-\frac{\overline{Y}}{\mu m} = \iint\left[u^2+v^2+2w(vx-uy)+w^2(x^2+y^2)\right](v+wx)\,dx\,dy,$$

the integration extending over the surface of the disc.

Putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$, then we get

$$-\frac{\overline{X}}{\mu m} = \int_0^{2\pi} \left[\tfrac{1}{2}u\,(u^2+v^2)\,r^2 + \tfrac{1}{4}\,w^2 u r^4 + \tfrac{1}{2}\,w^2 u \sin^2\theta r^4\right] d\theta,$$

where $r^2\left(\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}\right) = 1$, since evidently $\displaystyle\int_0^{2\pi} r^3\cos\theta\,d\theta = 0$,

$$\int_0^{2\pi} r^3 \sin\theta\,d\theta = 0, \quad \int_0^{2\pi} r^5 \sin\theta\,d\theta = 0, \quad \int_0^{2\pi} \sin\theta\cos\theta\,r^4\,d\theta = 0.$$

Performing the integration with respect to θ, and putting $\pi m ab = \mathrm{M}$ = mass of disc, we have $\overline{X} = -\mu\mathrm{M}\left[u\,(u^2+v^2) + \tfrac{1}{4}w^2 u\,(a^2+3b^2)\right]$, and similarly for $\overline{Y}$, hence $\mathrm{M}\dfrac{du}{dt} = \overline{X} = \&c.$, $\mathrm{M}\dfrac{dv}{dt} = \&c.$

2. If $a = b$, or the disc be circular, we have $v\dfrac{du}{dt} - u\dfrac{dv}{dt} = 0$, hence the path of the centre is a straight line.

7878. (By R. Knowles, B.A.)—In any triangle, prove that

$$\mathrm{E} \equiv \frac{a^2\cot\frac{1}{2}A + b^2\cot\frac{1}{2}B + c^2\cot\frac{1}{2}C}{a^2\tan\frac{1}{2}A + b^2\tan\frac{1}{2}B + c^2\tan\frac{1}{2}C} = \frac{R+r}{R-r}.$$

Solution by G. G. Storr, B.A.; Â. Mukhopâdhyây; *and others.*

We have $$\frac{\mathrm{E}+1}{\mathrm{E}-1} = \frac{a^2(\cot\frac{1}{2}A + \tan\frac{1}{2}A) + \dots + \dots}{a^2(\cot\frac{1}{2}A - \tan\frac{1}{2}A) + \dots + \dots}$$

$$= \frac{\dfrac{2a^2}{\sin A} + \dots + \dots}{\dfrac{2a^2\cos A}{\sin A} + \dots + \dots} = \frac{2\,\dfrac{a}{\sin A}(a+b+c)}{2\,.\,\dfrac{a}{\sin A}(a\cos A + b\cos B + c\cos C)} = \frac{2s}{a\cos A + \dots + \dots}$$

$$= \frac{2s}{\dfrac{a}{2\sin A}\,.\,(\sin 2A + \sin 2B + \sin 2C)} = \frac{4s\,.\,\sin A}{4a\sin A\sin B\sin C} = \frac{s}{a\sin B\sin C}$$

$$= \frac{s}{a\,.\,\dfrac{2\Delta}{ac}\,.\,\dfrac{2\Delta}{ba}} = \frac{abc\,.\,s}{4\Delta^2} = \frac{R}{r}, \quad \text{therefore} \quad \mathrm{E} = \frac{R+r}{R-r}.$$

6903. (By W. J. C. Sharp, M.A.)—Find the law of density, varying as the depth, that the centre of pressure of a semicircle, just immersed with its diameter horizontal, may divide the vertical radius as $m : n$.

Solution by the Proposer; E. Rutter; *and others.*

If ρ denote the density at any point of the fluid, a function of the depth, and $\bar{x}$ the depth of the centre of pressure of any curve bounded by the surface, $\bar{x}\int_0^a y\rho\, dx = \int_0^a xy\rho\, dx$, a being the abscissa of the lowest point of the curve.

If then $\bar{x} = \frac{m}{m+n} a$ and $y\rho = \phi''(x)$, the above equation becomes

$$\frac{m}{m+n} a\,[\phi'(a) - \phi'(0)] = a\,\phi'(a) - \phi(a) + \phi(0),$$

or
$$\left(\frac{m}{n}+1\right)\phi(a) + a\,\phi'(a) + \frac{m}{n} la + \left(\frac{m}{n}+1\right) l' = 0;$$

therefore
$$a^{-\left(\frac{m}{n}+1\right)}\phi(a) - la^{-\frac{m}{n}} - l'a^{-\left(\frac{m}{n}+1\right)} = l'',$$

and $\phi(a) = la + l' + l''a^{\left(\frac{m}{n}+1\right)}$ $\therefore$ $y\rho = \phi''(x)\, \alpha\, x^{\frac{m-n}{n}}$ and $\rho\, \alpha\, \frac{x^{\frac{m-n}{n}}}{y}$

Thus, for the semicircle with the diameter (1) in the surface, and (2) with the diameter downwards, $\rho\, \alpha\, \frac{x^{\frac{m-n}{n}}}{[(a^2-x^2)]^{\frac{1}{2}}}$, $\rho\, \alpha\, \frac{x^{\frac{m-n}{n}}}{(2ax-x^2)^{\frac{1}{2}}}$.

On Homological Polar Reciprocal Curves.

By Maurice d'Ocagne.

Let C and C′ be two curves polar reciprocal with respect to a conic K. Let us take on C a point P and the tangent t at this point; let t' and P′ be the corresponding tangent and point on C′. If all the straight lines such as PP′ pass through the same point A, all the points such as tt' are situated on the same straight line a, the polar of A with respect to K; consequently the curves C and C′ are *homological*, the point A being the *centre* and the straight line a the *axis of homology*.

We propose to ascertain all the curves that are homological of their reciprocal polar with respect to a given conic K, the corresponding points being the same in both cases; or, in other words, the curves such that the straight lines that join two corresponding points, one on the curve, the other on the reciprocal polar, pass through the same point.

Let $f(X, Y, Z) = 0$ be the homogeneous equation of the directing conic K.

The polar t' of the point P (x, y, z) of the curve C has for its equation

$$xf'_X + yf'_Y + zf'_Z = 0.$$

The point P′ where this straight line touches its envelop C′ is given

by its intersection with the straight line $dxf'_X + dyf'_Y + dzf'_Z = 0$. The equation of the straight line PP′ will therefore be

$$\frac{dx f'_X + dy f'_Y + dz f'_Z}{x f'_X + y f'_Y + z f'_Z} = \frac{dx f'_x + dy f'_y + dz f'_z}{x f'_x + y f'_y + z f'_z},$$

or, in virtue of the theorem of homogeneous functions,

$$\frac{dx f'_X + dy f'_Y + dz f'_Z}{x f'_X + y f'_Y + z f'_Z} = \frac{dx f'_x + dy f'_y + dz f'_z}{2f(x, y, z)}.$$

For this straight line to pass through a fixed point (X_1, Y_1, Z_1) it is necessary that we should have, whatever may be the point (x, y, z) of the curve C,

$$\frac{dx f'_{X_1} + dy f'_{Y_1} + dz f'_{Z_1}}{x f'_{X_1} + y f'_{Y_1} + z f'_{Z_1}} = \frac{dx f'_x + dy f'_y + dz f'_z}{2f(x, y, z)}.$$

Let $f(x, y, z) = S$, $x f'_{X_1} + y f'_{Y_1} + z f'_{Z_1} = M$; the preceding equation becomes $\frac{dM}{M} = \frac{dS}{2S}$; whence, integrating, $S - \lambda M^2 = 0$, λ being an arbitrary constant. We therefore find the equation of the conics bitangent to the directing conic K, the chord of contacts being the polar of the point (X_1, Y_1, Z_1) with respect to K.

Thus, *the only curves that are homological of their reciprocal polars, the corresponding points being the same in both cases, are the conics bitangent to the directing conic.*

The axis of homology is confounded with the chord of contacts, and the centre of homology with the pole of this chord.

In particular, it is known that the locus of the vertex of a constant angle (an isoptic curve) the sides of which are tangent to a parabola is an hyperbola bitangent to that parabola, the chord of contacts being the directrix of the parabola. Therefore, if H is an hyperbola isoptic of the parabola P, H′ the reciprocal polar of H with respect to P, the straight lines that join the points of H with the corresponding points of H′ all pass through the focus of P. We had, in an indirect way, obtained this theorem geometrically in one of our notes on the symmedian.

[See *Nouvelles Annales de Mathématiques*, Series 3, Vol. II., p. 462, § 26.]

NOTE ON QUESTION 6113 (Vol. XXXII., p. 107). *By* Professor CATALAN.

1. Dans les *Comptes Rendus* (tome LIV., p. 659), et plus tard dans les *Mélanges Mathématiques* (1868, p. 202), j'ai publié les formules suivantes :—

$$\pm S_{2k+1} = (2k+1)\left\{p^{k-1}q - \frac{(k-2)(k-3)}{2.3}p^{k-4}q^3 + \dots\right\},$$

$$\pm S_{2k} = 2p^k - 2k\left\{\tfrac{1}{2}(k-2)p^{k-3}q^2 - \frac{(k-3)(k-4)(k-5)}{2.3.4}p^{k-6}q^4 + \dots\right\},$$

relatives aux sommes des puissances $2k+1$ ou $2k$ des racines de $x^3+px+q=0$.

2. Au moyen de ces formules, on pourra former autant d'identités que l'on voudra, analogues à celle dont il s'agit ; il suffit, pour cela, d'éliminer p et q. Je trouve, par exemple :—

$$25\,S_3S_7 = 21\,S_5^2 \quad\ldots\ldots(1),$$

$$5^5 7^3 (S_5 S_7)^2 S_{11} = 11\left[5^7 S_7^5 + 7^5 S_5^7\right] \quad\ldots\ldots(2);$$

ou bien : $$25\,(a^3+b^3+c^3)\,(a^7+b^7+c^7) = 21\,(a^5+b^5+c^5)^2,$$

$$5^5\,7^3\,(a^5+b^5+c^5)^2\,(a^7+b^7+c^7)^2\,(a^{11}+b^{11}+c^{11})$$
$$= 11\left[5^7\,(a^7+b^7+c^7)^5 + 7^5\,(a^5+b^5+c^5)^7\right],$$

... ...

si $a+b+c=0$.

3. Si les coefficients p, q sont entiers, les sommes S_3, S_5, S_7, S_{11} sont respectivement, divisibles par 3, 5, 7, 11 (*loc. cit.*), faisant

$$S_3 = 3A_3,\quad S_5 = 5A_5,\quad S_7 = 7A_7,\quad S_{11} = 11A_{11}\quad\ldots\ldots(3),$$

on réduit, comme il suit, les relations (1), (2) :—

$$A_3\,A_7 = A_5^2,\quad (A_5\,A_7)^2\,A_{11} = A_5^7 + A_7^5 \quad\ldots\ldots(1', 2'),$$

... ...

4. D'après la dernière égalité, A_5^2 divise A_7^5, A_7^2 divise A_5^7.

Il serait facile d'étendre ces propriétés.

8218. (By Professor Steggall, M.A., F.R.S.E.)—Sixteen players draw for a tennis match: the winners draw and play again, and so on; prove that, of the last pair left in, one must be the best player, but the other may be only the ninth; and that the chances of the second, third, &c. being left in are as 3432 : 1716 : 792 : 330 : 120 : 36 : 8 : 1.

Solution by (1) Prof. Swaminatha Aiyar, B.A.; (2) D. Biddle.

1. No player can have any chance of being one of the two last that are left in, if he have not at least one inferior player not out, before drawing for partners in the third set of games, or at least three inferior players not out before drawing for the second set, or at least seven inferior players among the original sixteen. In other words, let once in the course of the contest, either at the beginning or before drawing for partners in the second set, or before drawing for the third set, the then surviving players be divided into two (numerically) equal parties, in all manner of ways, so as always to make a particular player P the best player of the party to which he belongs; if the players cannot be so divided, P has no chance of being a winner in the third set of games.

The number of different ways in which P shall be the best player of the party to which he belongs (the division of players into two equal parties taking place at the beginning) is ${}_pC_7$, where p is the number of players

inferior to P among the original sixteen. The number of different ways in which P shall be a winner in the first set, and the best player of the party to which he belongs in the second (the division taking place before drawing for the second set), is ${}_{p}C_7 \,.\, 7^2 \,.\, 5^2 \,.\, 3^2$. And the number of different ways in which P shall be a winner in the first two sets of games, and the best player of the party to which he belongs in the third set (the division taking place before drawing for the third set), is ${}_{p}C_7 \,.\, 7^2 \,.\, 5^2 \,.\, 3^2 \,.\, 3^2$. And the chances of P are proportional to this. Therefore the chances of the first nine players are as ${}_{15}C_7 : {}_{14}C_7 : {}_{13}C_7 :$ &c. $: {}_{7}C_7$.

Generally, if there be $2m$ players where $m = 2^n$, the chances are as

$$ {}_{2m-1}C_{m-1} : {}_{2m-2}C_{m-1} : \text{\&c.} : {}_{m-1}C_{m-1}. $$

2. *Otherwise* :—Representing the winners by Roman numerals, we can easily select the pairs as follows :—

$$ (1)\ \mathrm{I}_2,\ \mathrm{III}_4,\ \mathrm{V}_6,\ \mathrm{VII}_8,\ \mathrm{IX}_{13},\ \mathrm{XI}_{12},\ \mathrm{XIII}_{14},\ \mathrm{XV}_{16}; $$

$$ (2)\ \mathrm{I}_3,\ \mathrm{V}_7,\ \mathrm{IX}_{11},\ \mathrm{XIII}_{15};\quad (3)\ \mathrm{I}_5,\ \mathrm{IX}_{13};\quad (4)\ \mathrm{I}_9; $$

in which I. is necessarily the best player, having beaten all the others, either directly or indirectly; but in which the last to be vanquished is the ninth. That which is requisite for this contingency, is that, until the final contest, the eight best players shall pair among themselves, and likewise the eight worst. No player below the ninth can possibly be a winner more than twice, for simple lack of inferiors to encounter.

The respective chances of the nine best men being left in to the final, may be represented as follows. Let

$$ b = \tfrac{6}{7} \cdot \tfrac{2}{3},\quad c = \tfrac{5}{7}\left(\tfrac{1}{5} \cdot \tfrac{2}{3} + \tfrac{4}{5} \cdot \tfrac{1}{3}\right),\quad d = \tfrac{4}{7}\left(\tfrac{2}{5} \cdot \tfrac{1}{3} + \tfrac{3}{5} \cdot \tfrac{1}{3} \cdot \tfrac{1}{3}\right),\quad e = \tfrac{3}{7} \cdot \tfrac{3}{5} \cdot \tfrac{1}{3} \cdot \tfrac{1}{3}. $$

Then $\quad P_1 = \tfrac{15}{15} \cdot \tfrac{7}{7} \cdot \tfrac{3}{3} = \tfrac{6435}{6435}, \qquad P_2 = \tfrac{14}{15} b = \tfrac{3432}{6435},$

$$ P_3 = \tfrac{13}{15}\left(\tfrac{1}{13} b + \tfrac{12}{13} c\right) = \tfrac{1716}{6435}, \quad P_4 = \tfrac{12}{15}\left[\tfrac{2}{13} c + \tfrac{11}{13}\left(\tfrac{1}{11} c + \tfrac{10}{11} d\right)\right] = \tfrac{792}{6435}, $$

$$ P_5 = \tfrac{11}{15}\left\{\tfrac{3}{13}\left(\tfrac{1}{11} c + \tfrac{10}{11} d\right) + \tfrac{10}{13}\left[\tfrac{2}{11} d + \tfrac{9}{11}\left(\tfrac{1}{9} d + \tfrac{8}{9} e\right)\right]\right\} = \tfrac{330}{6435}, $$

$$ P_6 = \tfrac{10}{15}\left\{\tfrac{4}{13}\left[\tfrac{2}{11} d + \tfrac{9}{11}\left(\tfrac{1}{9} d + \tfrac{8}{9} e\right)\right] + \tfrac{9}{13}\left[\tfrac{3}{11}\left(\tfrac{1}{9} d + \tfrac{8}{9} e\right) + \tfrac{8}{11} \cdot \tfrac{3}{9} e\right]\right\} = \tfrac{120}{6435}, $$

$$ P_7 = \tfrac{9}{15}\left\{\tfrac{5}{13}\left[\tfrac{3}{11}\left(\tfrac{1}{9} d + \tfrac{8}{9} e\right) + \tfrac{8}{11} \cdot \tfrac{3}{9} e\right] + \tfrac{8}{13}\left[\tfrac{4}{11} \cdot \tfrac{3}{9} e + \tfrac{7}{11} \cdot \tfrac{3}{9} \cdot \tfrac{1}{7} e\right]\right\} = \tfrac{36}{6435}, $$

$$ P_8 = \tfrac{8}{15}\left\{\tfrac{6}{13}\left[\tfrac{4}{11} \cdot \tfrac{3}{9} e + \tfrac{7}{11} \cdot \tfrac{3}{9} \cdot \tfrac{1}{7} e\right] + \tfrac{7}{13}\left[\tfrac{5}{11} \cdot \tfrac{3}{9} \cdot \tfrac{1}{7} e\right]\right\} = \tfrac{8}{6435}, $$

$$ P_9 = \tfrac{7}{15}\left\{\tfrac{7}{13}\left[\tfrac{5}{11} \cdot \tfrac{3}{9} \cdot \tfrac{1}{7} e\right]\right\} = \ldots\ldots \tfrac{1}{6435}. $$

Here it is evident that b, c, d, e represent his remaining chance, when the player, winning in the primary encounter, is second, third, fourth, fifth, in order of merit, of those pairing for the next round. The fractions attached show his chance of securing such relative position, provided the sum of the products having reference to the given letter be taken, when the letter occurs more than once.

[Let the chance that the 9th player is left in at the last round be called p; then this requires that the last 8 players (9 to 16) should be always drawn against each other. The chance that any other player is left in is $p \times$ the number of groups consisting of himself and 7 inferior players; for each such group affords the same chance (p) that the members of it will be drawn only against each other. This number clearly is the number of combinations of the players inferior to him, 7 at a time. For example, the chance that the fifth is left in is ${}_{11}C_7 \times p = 330p$; and that the first is left in is ${}_{15}C_7 \times p = 6435p = 1$. Hence the result, and the value of p. This last may be obtained directly thus :—The chance that

the last 8 are drawn against one another in the first round is $\frac{7}{15} \times \frac{5}{13} \times \frac{3}{11} \times \frac{1}{9}$; the chance that the 4 winners are drawn against each other in the second round is $\frac{3}{7} \times \frac{1}{5}$; and the chance that the 2 winners are drawn against each other in the third round is $\frac{1}{3}$. This gives as the chance that the 9th is left in

$$p = \frac{7.5.3.1.3.1.1}{15.13.11.9.7.5.3.1} = \frac{1}{6435}.$$

The extension to 2^n players is obvious, as is also the unfairness of awarding the second prize.]

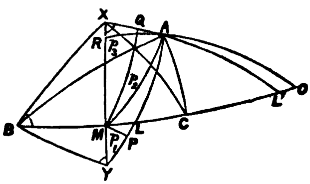
7814. (By the EDITOR.)—If p_1, p_2 be the perpendiculars drawn from the mid-point M of the base of a spherical triangle on the great circle bisectors of the vertical angle A, and p_3 the perpendicular from A on the great circle perpendicular to the base through M, prove that (1)

$$\sin p_1 \sin p_2 = \tfrac{1}{2} \sin p_3 \sin \tfrac{1}{2}a \sin (B + C),$$

and (2) the analogous theorem for a plane triangle is $4p_1 p_2 = ap_3 \sin A$.

Solutions by D. EDWARDES; W. J. MCCLELLAND, M.A.; *and others.*

1. Let the angle bisectors meet the base in L, L′; and let RA $(= p_3)$ meet the base produced in O; then O is the pole of MR or XY. From the triangles ACO, BAO, we have



$$\cos b = \sin p_3 \sin \tfrac{1}{2}a + \cos p_3 \cos \tfrac{1}{2}a \cos O,$$

$$\cos c = -\sin p_3 \sin \tfrac{1}{2}a + \cos p_3 \cos \tfrac{1}{2}a \cos O,$$

therefore $\qquad 2 \sin p_3 \sin \tfrac{1}{2}a = \cos b - \cos c.$

Again, $\qquad \dfrac{\sin BL}{\sin c} = \dfrac{\sin CL}{\sin b} = \dfrac{2 \cos \frac{1}{2}a \sin ML}{\sin c - \sin b} = \dfrac{\sin \frac{1}{2}A}{\sin ALC},$

where, from the triangle MPL, $\sin p_1 = \sin ML \sin ALC$,

therefore $\qquad 2 \sin p_1 \cos \tfrac{1}{2}a = (\sin c - \sin b) \sin \tfrac{1}{2}A.$

In the same way, $2 \sin p_2 \cos \tfrac{1}{2}a = (\sin c + \sin b) \cos \tfrac{1}{2}A,$

therefore $\quad 4 \sin p_1 \sin p_2 \cos^2 \tfrac{1}{2}a = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \sin (c + b) \sin (c - b),$

therefore $\quad \dfrac{2 \sin p_1 \sin p_2 \cos^2 \frac{1}{2}a}{\sin p_3 \sin \frac{1}{2}a} = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \cos \tfrac{1}{2}(b + c) \cos \tfrac{1}{2}(b - c)$

$$= 2 \cos \tfrac{1}{2}(B + C) \sin \tfrac{1}{2}(B + C) \cos^2 \tfrac{1}{2}a,$$

and therefore $\qquad \sin p_1 \sin p_2 = \tfrac{1}{2} \sin p_3 \sin \tfrac{1}{2}a \sin (B + C).$

2. In the plane triangle, we have

$$ML = \tfrac{1}{2} \frac{a(c-b)}{c+b}, \quad 2p_1 = \frac{a(c-b)}{c+b} \sin (B + \tfrac{1}{2}A),$$

$$ML' = \tfrac{1}{2} \frac{a(c+b)}{c-b}, \quad 2p_2 = \frac{a(c+b)}{c-b} \cos (B + \tfrac{1}{2}A), \quad p_3 = \tfrac{1}{2}a - b \cos C = \frac{c^2 - b^2}{2a},$$

so that $$\frac{2p_1p_2}{p_3} = a^3 \frac{\sin(B+\frac{1}{2}A)\cos(B+\frac{1}{2}A)}{c^2-b^2},$$

or $$\frac{4p_1p_2}{p_3} = a^3 \frac{\sin(C-B)}{c^2-b^2} = a \frac{\sin^2 A \sin(C-B)}{\sin(C+B)\sin(C-B)},$$

that is, $$4p_1 p_2 = ap_3 \sin A.$$

[*Otherwise*:—Drawing lines as in the figure, and putting AM $= \lambda$,
we have $\sin p_1 = \sin\lambda \sin \text{MAP}$, $\sin p_2 = \sin\lambda\cos\text{MAP}$,
therefore $2\sin p_1 \sin p_2 = \sin^2\lambda \sin 2\text{MAP} = \sin^2\lambda\sin(\text{BAL}-\text{CAL})$...($\alpha$).
But $\sin\lambda\sin\text{BAL} = \sin\frac{1}{2}a\sin B$, and $\sin\lambda\sin\text{CAL} = \sin\frac{1}{2}a\sin C$,
therefore $\sin^2\lambda(\sin^2\text{BAL}-\sin^2\text{CAL}) = \sin^2\frac{1}{2}a(\sin^2 B - \sin^2 C)$,
or $\sin^2\lambda\sin A\sin(\text{BAL}-\text{CAL}) = \sin^2\frac{1}{2}a\sin(C+B)\sin(C-B)$ ...(β).
Again, we have $\sin p_3 = \sin\lambda\cos\text{AMC}$,
therefore $\sin\frac{1}{2}a\sin p_3 = \cos b - \cos\lambda\cos\frac{1}{2}a = \cos b - \frac{1}{2}(\cos b + \cos c)$,
by a known formula (see Davies' *Hutton*, Vol. II., p. 69),

$$= \tfrac{1}{2}(\cos b - \cos c) = \sin\tfrac{1}{2}(c+b)\sin\tfrac{1}{2}(c-b) = \frac{\sin(C-B)}{\sin A}\sin^2\tfrac{1}{2}a \quad ...(\gamma),$$

by Gauss's *Analogies*.
From (α), (β), (γ), we get the result stated in (1).

Or, again, observing that $\angle \text{XBL} = \frac{1}{2}(B+C)$, we have

$$\frac{\sin p_1 \sin p_2}{\sin p_3} = \frac{\sin\text{MX}\sin\text{MY}\sin X\sin Y}{\sin\text{AX}\sin X} = \frac{\sin\text{MX}\sin\text{MY}\sin Y}{\sin\text{AX}}$$

$$= \frac{\sin\text{MX}\sin\text{MY}}{\sin\text{XY}} = \sin\text{MY}\sin Y\sin\tfrac{1}{2}(B+C) = \tfrac{1}{2}\sin\tfrac{1}{2}a\sin(B+C).$$

If we consider the radius of the sphere to be infinite, the spherical surface becomes a plane; and, since an evanescent arc is equal to its sine, the plane theorem in (2) follows immediately. See McClelland's *Spherical Trigonometry* Part I., p. 38.]

7810. (By Professor Hudson, M.A.)—Three equal fine straight tubes, equally inclined to the vertical, meet at a point where there is free communication between them; equal volumes of three different liquids are poured, one into each: obtain a condition that equilibrium may be possible with the whole of each liquid continuous, and in this case determine the possible positions of the common surfaces.

Solution by D. Biddle.

Let A, B, C be the three non-miscible fluids (it is enough that one, the heaviest, refuses to mix with either of the others); let a, b, c be their several specific gravities, and let 1 = the length of tube occupied by the volume of each. A, the heaviest, must be poured in first. It will rise in each tube to a distance $= \frac{1}{3}$ from the junction. C, the lightest, must next be poured in gradually. If $c < \frac{1}{2}a$, the whole of C may be poured in with-

out risk of breaking its continuity, because A will still occupy a portion of the tube down which C is poured, though pushed before it and rising to an equal height in the other two tubes. But, if $c > \frac{1}{2}a$, a portion of C must be reserved until after B is poured into its proper tube, a matter of easy adjustment. A portion of A continues to occupy each tube. Let x = portion of A in B's tube, and y = portion of A in C's tube; then

$$b + ax = c + ay = (1 - x - y)\,a,$$

whence $x = \dfrac{a+c-2b}{3a}$, and $y = \dfrac{a+b-2c}{3a}$, and $1 - x - y = \dfrac{a+b+c}{3a}$.

These, therefore, are the distances (from the junction) of the three surfaces of A, unity being the length of tube occupied by B or C.

N.B.—It is requisite that $2b < (a+c)$.

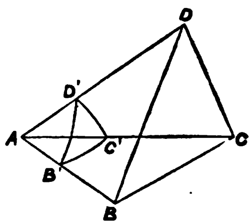
7570. (By Professor Wolstenholme, M.A., Sc.D.)—In a tetrahedron ABCD, the lengths of the edges DA, DB, DC are a, b, c, and those of the respectively opposite edges BC, CA, AB are $a+x$, $b+x$, $c+x$; prove that (1) if the sum of the dihedral angles at A, B, or C be denoted by $180° + 2S$,

$$\cos S = \frac{4abc + (b+c-a+x)(c+a-b+x)(a+b-c+x) + 2x\,(a+b+c+x)^2}{(a+b+c+x)\left[(a+b+c+3x)(b+c-a+x)(c+a-b+x)(a+b-c+x)\right]^{\frac{1}{2}}};$$

(2) if the sum of the dihedral angles at D be $180° + 2S'$,

$$\cos S' = \frac{(a+b+c+x)\left[4\,(bc+ca+ab) - (a+b+c)\,(a+b+c+x)\right] - 4abc}{\left[(a+b+c+x)^3\,(b+c-a-x)(c+a-b-x)(a+b-c-x)\right]^{\frac{1}{2}}}.$$

Solution by B. Hanumanta Rau, B.A.; Professor Nash, M.A.; *and others.*



1. The edges DB + BC = DA + AC,

therefore $\widehat{DB} + \widehat{BC} = \widehat{DA} + \widehat{AC}$,

where $\widehat{DB}$ represents the angle between the faces meeting in DB. Adding $\widehat{BA}$ to both sides, the sum of the dihedral angles at B is equal to those at A and also those at C.

With A as centre describe a sphere meeting the faces in the arcs D′B′, B′C′, and and C′D′. The angles D′, B′, C′ of the spherical triangle are the dihedral angles $\widehat{AD}$, $\widehat{AB}$, and AC; therefore

$$180° + 2S = D' + B' + C', \quad \text{therefore} \quad \cos S = \sin \tfrac{1}{2}(D' + B' + C').$$

But, by spherical trigonometry, $\frac{1}{4}\cos^2(D' + B' + C')$

$$= \frac{1 - \cos^2 B'C' - \cos^2 C'D' - \cos^2 D'B' + 2\cos B'C' \cos C'D' \cos D'B'}{2\,(1 + \cos B'C')(1 + \cos C'D')(1 + \cos D'B')},$$

therefore $\sin^2 \frac{1}{2}(D' + B' + C') = \dfrac{(1 + \cos B'C' + \cos C'D' + \cos D'B')^2}{2\,(\ldots)(\ldots)(\ldots)}$.

Again, $\cos B'C' = \cos BAC = \dfrac{(b+x)^2+(c+x)^2-(a+x)^2}{2(b+x)(c+x)}$,

with similar values for $\cos C'D'$ and $\cos D'B'$. Substituting these values and clearing of fractions, we have $\cos S = \dfrac{\text{Numerator}}{\text{Denominator}}$, where

$$\begin{aligned}\text{Numerator} &= 2a(b+x)(c+x)+a\left[(b+x)^2+(c+x)^2-(a+x)^2\right] \\ &\quad +(c+x)\left[a^2+(b+x)^2-c^2\right]+(b+x)\left[a^2+(c+x)^2-b^2\right] \\ &= 2x^3+3(a+b+c)x^2+\underset{\smile}{4}(bc+ca+ab)x \\ &\quad +2abc+a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2),\end{aligned}$$

which agrees with the numerator in the question, if the last coefficient 2 be omitted,

$$\begin{aligned}(\text{Denominator})^2 &= \left[(b+c+2x)^2-(a+x)^2\right][\quad][\quad] \\ &= (a+b+c+3x)(b+c-a+x)(\quad)(\quad)(\quad)(\quad) \\ &= (a+b+c+x)^2(a+b+c+3x)(b+c-a+x)(\quad)(\quad).\end{aligned}$$

2. As in the first part,

$$\begin{aligned}\cos S' &= \frac{1+\cos ADB+\cos BDC+\cos CDA}{\left[2(1+\cos ADB)(1+\cos BDC)(1+\cos CDA)\right]^{\frac12}} \\ &= \frac{2abc+c\left[a^2+b^2-(c+x)^2+a\left[b^2+c^2-(a+x)^2\right]+b\left[a^2+c^2-(b+x)^2\right]\right.}{\left\{\left[(a+b)^2-(c+x)^2\right]\left[(b+c)^2-(a+x)^2\right]\left[(a+c)^2-(b+x)^2\right]\right\}^{\frac12}} \\ &= \frac{(a+b+c+x)\left[c(a+b-c-x)+a(b+c-a-x)+b(a+c-b-x)\right]-4abc}{\left[(a+b+c+x)^3(a+b-c-x)(b+c-a-x)(a+c-b-x)\right]^{\frac12}}\end{aligned}$$

The numerator can be put in the form

$$(a+b+c+x)\left[4(bc+ca+ab)-(a+b+c)(a+b+c+x)\right]-4abc.$$

7936. (By D. Biddle.)—A circular target is provided with a rectangular impenetrable screen, of just sufficient depth to cover it. This screen is raised so as fully to expose the target, and is then allowed to fall, like the blade of a guillotine, until the target is again fully exposed. During the descent, which is accomplished with the usual accelerated motion, a bullet making for the target reaches the combined structure. Find the probability that the bullet hits the target and not the screen.

Solution by the Rev. T. C. Simmons, M.A.

Taking unity to denote the radius of the target, let x, y be the coordinates of any point thereon referred to horizontal and vertical axes through the centre. Then, if $8^{\frac12}g^{-\frac12}=\tau$, the point will be exposed during an interval represented by $\frac12\tau(1-y)^{\frac12}$, then hidden, then exposed again during an interval $\tau-\frac12\tau(3-y)^{\frac12}$. Whence, since τ denotes the whole time under consideration, the chance of the point being exposed at any random instant is $1-\frac12(3-y)^{\frac12}+\frac12(1-y)^{\frac12}$. So that, if the bullet aims at any

elementary horizontal strip $2x\,dy$ or $2(1-y^2)^{\frac{1}{2}}\,dy$, the chance of its hitting the screen is $\frac{1}{2}(3-y)^{\frac{1}{2}}-\frac{1}{2}(1-y)^{\frac{1}{2}}$, and the whole chance taken throughout the target will consequently be

$$\frac{1}{\pi}\int_{-1}^{1}\left[(3-y)^{\frac{1}{2}}-(1-y)^{\frac{1}{2}}\right](1-y^2)^{\frac{1}{2}}\,dy,$$

or $$\frac{1}{\pi}\int_{-1}^{1}(3+2y-y^2)^{\frac{1}{2}}(1-y)^{\frac{1}{2}}\,dy-\frac{1}{\pi}\int_{-1}^{1}(2-1-y)(1+y)^{\frac{1}{2}}\,dy.$$

Putting $y=1-2u^{\frac{1}{2}}$ in the first integral, and $y=v-1$ in the second, this becomes

$$\frac{2^{\frac{3}{2}}}{\pi}\int_0^1 u^{-\frac{1}{4}}(1-u)^{\frac{1}{2}}\,du-\frac{1}{\pi}\int_0^2\left(2v^{\frac{1}{2}}-v^{\frac{3}{2}}\right)dv$$

$$=\frac{2^{\frac{3}{2}}}{\pi}\cdot\frac{\Gamma\left[\frac{3}{4}\right].\Gamma\left[\frac{3}{2}\right]}{\Gamma\left[\frac{9}{4}\right]}-\frac{1}{\pi}\left[\tfrac{4}{3}.2^{\frac{3}{2}}-\tfrac{2}{5}.2^{\frac{5}{2}}\right]$$

$$=\frac{16\sqrt{2}}{15\pi}\left\{\frac{2\Gamma[1\cdot75].\Gamma[1\cdot5]}{\Gamma[1\cdot25]}-1\right\}=\frac{16\sqrt{2}}{15\pi}[1\cdot797212-1].$$

Calculating by logarithms, we find this reduces to ·382796, which is the chance that the bullet will hit the screen. Hence the required chance of hitting the target is ·617204, a result which ought to be reliable to at least four places of decimals.

It may be of interest to note that as the solution is unaffected by the substitution of $g\cos\alpha$ or f for g, the same result will hold when the screen and target are in a plane inclined to the vertical, and even when there is friction, provided it acts uniformly. Nor will the substitution of an elliptic for a circular screen make any difference, since in this case the length of any horizontal strip bears a fixed proportion to the corresponding horizontal strip of a circle, whose horizontal tangents coincide with those of the ellipse.

8030. (By R. Knowles, B.A.)—A circle through the focus and the ends of a normal chord of the parabola $y^2=4ax$, meets the parabola again in PQ; prove that (1) the locus of the pole of PQ with respect to the parabola, and (2) its envelope, are respectively the parabolas

$$y^2+a(x+a)=0 \text{ and } y^2+16a(x-a)=0.$$

Solution by B. Hanumanta Rau, B.A.; E. Rutter; *and others.*

The normal at any point $(am^2, 2am)$ of the parabola is

$$y+mx=2am+am^3.$$

The equation to PQ must be the form $y-mx+k=0$, since the normal and PQ are equally inclined to the axis. Hence the equation to the circle is of the form $\lambda(y^2-4ax)+(y+mx-2am-am^3)(y-mx+k)=0.$

Equating the coefficients of x^2 and y^2, and remembering that the circle

passes through the focus $(0, a)$, we get $k \mp \frac{4a}{m} + ma = 0$. Thus PQ is

$$y - mx + \frac{4a}{m} + ma = 0 \quad \text{.............................(1).}$$

If $(x'y')$ is the pole of PQ, (1) must be equivalent to

$$yy' = 2a(x + x'), \text{ therefore } \frac{1}{y'} = \frac{m}{2a} = -\left(\frac{2}{m} + \frac{m}{2}\right)\frac{1}{x'}.$$

Eliminating m and suppressing the accents, we have the locus

$$y^2 + a(x + a) = 0.$$

Again, differentiating (1) with respect to m,

$$-x - \frac{4a}{m^2} + a = 0 \quad \text{or} \quad a - x = \frac{4a}{m^2}.$$

From (1), $\quad y^2 = m^2\left(x - a - \frac{4a}{m^2}\right)^2 = \frac{4a}{a - x}(2x - 2a)^2 = 16(a - x)a,$

or $\quad y^2 + 16a(x - a) = 0$, the envelope of (1).

7318. (By W. J. C. Sharp, M.A.)—Show that the tangents from the point $(x_1 y_1 z_1)$ on the non-singular cubic $x^3 + y^3 + z^3 + 6m\,xyz = 0$, touch at points of intersection of the conics $x_1 x^2 + y_1 y^2 + z_1 z^2 = 0$ and $\frac{x_1}{x} + \frac{y_1}{y} + \frac{z_1}{z} = 0$; and hence determine (1) the equation to the satellite of the line $ax + \beta y + \gamma z = 0$; (2) the quartic curve which meets the cubic at the points of contact of tangents drawn from the intersections of $ax + \beta y + \gamma z = 0$ with the curve.

Solution by the Proposer.

The coordinates of the tangential of (x, y, z) are (Salmon's *Higher Plane Curves*, p. 153), as $x(y^3 - z^3) : y(z^3 - x^3) : z(x^3 - y^3)$; therefore

$$kx_1 = x(y^3 - z^3), \quad ky_1 = y(z^3 - x^3), \quad kz_1 = z(x^3 - y^3),$$

and consequently $\quad x^2 x_1 + y^2 y_1 + z^2 z_1 = 0, \quad$ and $\quad \frac{x_1}{x} + \frac{y_1}{y} + \frac{z_1}{z} = 0.$

If $ax + \beta y + \gamma z = 0$, these three equations give

$$\beta z_1 y^2 + (\beta y_1 + \gamma z_1 - a x_1)\,yz + \gamma y_1 z^2 = 0,$$

and $$(a^2 y_1 + \beta^2{}_1 x)\,y^2 + 2\beta\gamma\, x_1 yz + (a^2 z_1 + \gamma^2 x_1)\,z^2 = 0,$$

the eliminant of which becomes (since $x_1^3 + y_1^3 + z_1^3 + 6m\,x_1 y_1 z_1 = 0$)

$$a^2 x_1 y_1 z_1 \left\{[a^4 - 2a(\beta^3 + \gamma^3) - 6m\,\beta^2\gamma^2]\,x_1 + [\beta^4 - 2\beta(\gamma^3 + a^3) - 6m\,\gamma^2 a^2]\,y_1 \right.$$
$$\left. + [\gamma^4 - 2\gamma(a^3 + \beta^3) - 6ma^2\beta^2]\,z_1\right\} = 0,$$

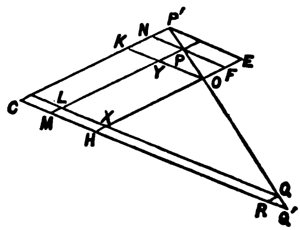
so that $\quad [a^4 - 2a(\beta^3 + \gamma^3) - 6m\,\beta^2\gamma^2]\,x + [\beta^4 - 2\beta(\gamma^3 + a^3) - 6m\gamma^2 a^2]\,y$
$$+ [\gamma^4 - 2\gamma(a^3 + \beta^3) - 6ma^2\beta^2]\,z = 0,$$

is the equation to the satellite of $\alpha x+\beta y+\gamma z=0$, a result agreeing with that given by Mr. W. R. ROBERTS, Quest. 5753, *Reprint*, Vol. XXXIII., p. 92.

Again, if $\alpha x_1+\beta y_1+\gamma z_1=0$, from this and the equations to the two conics $\begin{vmatrix} \alpha, & \beta, & \gamma \\ x^2, & y^2, & z^2 \\ yz, & zx, & xy \end{vmatrix}=0$, which represents a quartic which passes through the points of contact of all the tangents drawn to any cubic of the set $U+\lambda H=0$ from its intersections with the line $\alpha x+\beta y+\gamma z=0$. (See *Quarterly Journal*, Vol. XVI., p. 302.)

8208. (By the Rev. J. J. MILNE, M.A.)—Through a fixed point O a chord POQ of an hyperbola is drawn, and lines PL, QL are drawn parallel to the asymptotes. Show that the locus of L is a similar and similarly situated hyperbola.

Solution by the PROPOSER; PIERS C. WARD, M.A.; *and others.*



Since $LM=QR=P'N=EF$, therefore
$$LH=PE=\text{comp. } PK,$$
therefore
$$LO=OC-(MK+LH)$$
$$=OC-MN=\text{constant},$$
therefore $LX \cdot LY=\text{constant}$.

8086. (By Professor BYOMAKESA CHAKRAVARTI, M.A.)—A sportsman moves his gun round his shoulder at a fixed point so as to cover accurately a small bird which is moving uniformly, and fires when it is nearest him. Prove that he will miss, unless the scattering angle of his gun be greater than $2\sin^{-1} nv/V$, where v, V are the velocities of the bird and shot, the firing being supposed point-blank, and n the ratio of the distances of the bird from the muzzle of the gun and the shoulder.

Solution by D. EDWARDES.

The muzzle of the gun at the instant of firing is moving with velocity $(1-n)v$, which is the same as if the gun did not rotate, and the velocity of the bird was nv. Hence, if a shot is to hit the bird, its direction of flight must make an angle (θ) with the barrel such that $\sin\theta=nv/V$. The scattering angle must therefore be greater than $2\sin^{-1} nv/V$.

8205. (By W. S. M'Cay, M.A.)—If the cosines of the angles of a plane triangle be connected by the equation $\cos^2 A + \cos^2 B + \cos^2 C = \frac{3}{4}$; prove that the triangle *must* be equilateral.

Solution by Piers C. Ward, M.A.; J. O'Regan; *and others.*

We have $$2\cos A \cos B \cos (A+B) = -\tfrac{1}{4}.$$
Hence $$2\cos (A+B) + \cos (A-B) = (-1)^{\frac{1}{2}} \sin (A-B),$$
and $$2\cos (A+C) + \cos (A-C) = (-1)^{\frac{1}{2}} \sin (A-C);$$
therefore $\sin (A-B) = 0$ and $\sin (A-C) = 0$, therefore etc.

8203. (By Asûtosh Mukhopâdhyây, B.A., F.R.A.S.)—If
$$S_1 = \sin\theta + \tfrac{1}{3}\sin^3\theta + \tfrac{1}{5}\sin^5\theta + \dots \text{ ad inf.},$$
$$S_2 = \sin\theta - \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta - \dots \text{ ad inf.},$$
show that $S_1 = 2S_2$.

Solution by Rev. J. J. Milne, M.A.; P. C. Ward, M.A.; *and others.*

$$iS_1 = i\sin\theta - \tfrac{1}{3}(i\sin\theta)^3 + \dots = \tan^{-1}(i\sin\theta).$$
Let $$C_2 = \cos\theta - \tfrac{1}{3}\cos 3\theta + \dots$$
therefore $$C_2 + iS_2 = \epsilon^{\theta i} - \tfrac{1}{3}\epsilon^{3\theta i} + \dots = \tan^{-1}(\epsilon^{\theta i}),$$
$$C_2 - iS_2 = \epsilon^{-\theta i} - \tfrac{1}{3}\epsilon^{-3\theta i} + \dots = \tan^{-1}(\epsilon^{-\theta i}),$$
therefore $$2iS_2 = \tan^{-1}(\epsilon^{\theta i}) - \tan^{-1}(\epsilon^{-\theta i}),$$
therefore $$\tan(2iS_2) = \tfrac{1}{2}(\epsilon^{\theta i} - \epsilon^{-\theta i}) = i\sin\theta = \tan(iS_1), \quad \therefore S_1 = 2S_2.$$

8C65. (By Âsûtosh Mukhopâdhyây, B.A., F.R.A.S.)—If δ be the distance of the earth from the moon, κ the ratio of their masses, prove that the locus of a point, in the plane of the moon's orbit, where a body is equally attracted by the earth and the moon, is a circle of radius $\delta\sqrt{\kappa}/(1-\kappa)$.

Solution by D. Edwardes.

Let r, r' be the distances of the point from the centres of the moon and earth, M, m the masses of earth and moon, R the radius of the earth.

Then $$g\frac{m}{M}\cdot\frac{R^2}{r^2} = g\frac{R^2}{r^2} \text{ or } \frac{r}{r'} = \sqrt{\kappa}.$$

But the locus of a point the ratio of whose distances from two points distant δ apart is $\sqrt{\kappa}$, is a circle of radius $\delta\sqrt{\kappa}/1-\kappa$.

8189. (By Professor MATHEWS, M.A.)—Prove that the centre of the nine-point circle of a triangle ABC is the centre of inertia of three particles at A, B, C, whose masses are proportional to $(\sin 2B + \sin 2C)$, $(\sin 2C + \sin 2A)$, and $(\sin 2A + \sin 2B)$ respectively.

Solution by A. GORDON, M.A.; PIERS C. WARD, M.A.; *and others.*

The coordinates of the centre of gravity G of the particles would be given by $\alpha : \beta : \gamma = \cos(B-C) : \cos(C-A) : \cos(C-B)$; and the coordinates of the centre of the nine-point circle are $\frac{1}{2}(\alpha_1 + \alpha_2)$, $\frac{1}{2}(\beta_1 + \beta_2)$, &c., where $\alpha_1 = -2R\cos A + C\sin B$, $\alpha_2 = R\cos A$, &c., or the coordinates are $R\cos(B-C)$, &c., and hence coincide with the point G.

8021. (By H. L. ORCHARD, B.Sc., M.A.)—If the velocities of a heavy particle moving in a resisting medium be v and v' when the direction of motion is at angles of 45° with the horizon, show that the velocity at the highest point is $\dfrac{vv'}{(v^2+v'^2)^{\frac{1}{2}}}$, the resistance being supposed to vary as the *square* of the velocity.

Solutions by (1) *the* PROPOSER; (2) Rev. T. GALLIERS, M.A.

1. Since the velocity at any point of the path is $ue^{-ks}(1+p^2)^{\frac{1}{2}}$, the intrinsic equations for the two cases of 45° and for that of the highest point are (see TAIT and STEELE's *Dynamics*), respectively,

$$\tan\alpha\sec\alpha + \log(\tan\alpha + \sec\alpha) + fk^{-1}u^{-2} = 2^{\frac{1}{2}} + \log(2^{\frac{1}{2}}+1) + 2fk^{-1}v^{-2} \ldots (1),$$

$$= -2^{\frac{1}{2}} + \log(2^{\frac{1}{2}}-1) + 2fk^{-1}v'^{-2} \ldots\ldots (2),$$

$$= fk^{-1}r^{-2},\ r \text{ the velocity at the highest point} \ldots\ldots (3).$$

Hence, adding (1) and (2), we have, by means of (3),

$$2fk^{-1}(v^{-2}+v'^{-2}) = 2fk^{-1}r^{-2} \quad\text{or}\quad v^{-2}+v'^{-2} = r^{-2},$$

where the required result follows.

2. *Otherwise*:—Let V_1 and V_2 be the velocities at the two points where the direction of motion of the particle makes an angle ϕ with the horizon, then we may show that

$$\frac{1}{V_1} + \frac{1}{V_2^2} = \frac{2\cos^2\phi}{V^2} \ldots\ldots (1),$$

where V is the velocity at the highest point. For (see TAIT and STEELE's *Dynamics*, 3rd edition, Art. 236) the intrinsic equation to the path of the particle will be

$$p(1+p^2)^{\frac{1}{2}} + \log[p + (1+p^2)^{\frac{1}{2}}] + \frac{f}{ku^2}e^{2ks} = \tan\alpha\sec\alpha + \log(\tan\alpha + \sec\alpha) + \frac{f}{ku^2} \ldots\ldots (2),$$

$\tan\alpha$ being the value of p when $s = 0$; also

$$\left(\frac{ds}{dt}\right)^2 = v^2 = u^2 e^{-2ks}(1+p^2),$$

u being initial velocity parallel to axis of x; therefore (2) may be written

$$p(1+p^2)^{\frac{1}{2}}+\log\left[p+(1+p^2)^{\frac{1}{2}}\right]+\frac{f}{k}\cdot\frac{1+p^2}{v^2}$$
$$=\tan\alpha\sec\alpha+\log(\tan\alpha+\sec\alpha)+\frac{f}{ku^2}\;\ldots\ldots\ldots(3).$$

In this equation put p successively equal to $\tan\phi$ and $\tan(\pi-\phi)$, when, by the question, v will be V_1 and V_2 respectively. Therefore

$$\tan\phi\sec\phi+\log(\sec\phi+\tan\phi)+\frac{f}{k}\cdot\frac{1+\tan^2\phi}{V_1^2}$$
$$=\tan\alpha\sec\alpha+\log(\tan\alpha+\sec\alpha)+\frac{f}{ku^2}\;\ldots\ldots\ldots(4),$$

$$-\tan\phi\sec\phi+\log(\sec\phi-\tan\phi)+\frac{f}{k}\cdot\frac{1+\tan^2\phi}{V_2^2}$$
$$=\tan\alpha\sec\alpha+\log(\tan\alpha+\sec\alpha)+\frac{f}{ku^2}\;\ldots\ldots\ldots(5);$$

also when $p=0$, in (3), $v=V$, therefore

$$\frac{f}{k}\cdot\frac{1}{V^2}=\tan\alpha\sec\alpha+\log(\tan\alpha+\sec\alpha)+\frac{f}{ku^2}\;\ldots\ldots\ldots\ldots\ldots(6).$$

From (4) and (5), by addition,

$$\frac{f}{k}\sec^2\phi\left(\frac{1}{V_1^2}+\frac{1}{V_2^2}\right)=2\left\{\tan\alpha\sec\alpha+\log(\tan\alpha+\sec\alpha)+\frac{f}{ku^2}\right\}$$
$$=2\frac{f}{k}\cdot\frac{1}{V^2},\text{ by (6)},$$

since $\log(\sec\phi+\tan\phi)+\log(\sec\phi-\tan\phi)=\log(\sec^2\phi-\tan^2\phi)=0$,

therefore $$\frac{1}{V_1^2}+\frac{1}{V_2^2}=\frac{2\cos^2\phi}{V^2}.$$

[Quest. 8021 is the particular case obtained by making $\phi=\frac{1}{4}\pi$.]

8109. (By D. Edwardes,)—Prove that

$$u\equiv\int_0^\infty\frac{(\sin mx-\sin nx)^2}{x^2}\,dx=\frac{\pi}{2}(m\sim n).$$

Solution by A. Gordon, M.A.; Professor Nash, M.A.; *and others.*

$$\frac{du}{dm}=\int_0^\infty\frac{2\cos mn(\sin mx-\sin nx)}{x}$$
$$=\int_0^\infty\frac{\sin 2mx-\sin(n-m)x-\sin(n+m)x}{x}\,dx;$$

but $$\int_0^\infty\frac{\sin rx}{x}=\pm\frac{\pi}{2}\text{ as }r\text{ is }\pm;\text{ hence }\frac{du}{dm}=-\frac{\pi}{2},$$

therefore $u=-\frac{1}{2}\pi m+\phi(n)$, supposing m and n both positive and $n>m$; similarly $u=\frac{1}{2}\pi n+\phi(m)$; $\therefore\; u=\frac{1}{2}\pi(n-m)+$ a constant ($=0$, since u vanishes when $n=m$); therefore result is $u=\frac{1}{2}\pi(n\sim m)$.

8340. (By F. Morley, B.A.)—Show that (1) on a chess-board the number of squares visible is 204, and the number of rectangles (including squares) visible is 1,296; and (2) on a similar board, with n squares in each side, the number of squares is the sum of the first n square numbers, and the number of rectangles (including squares) is the sum of the first n cube numbers.

Solution by Professor Cayley, F.R.S.

In a board of n^2 squares, the number of pairs of vertical lines at a distance from each other of $n-r+1$ squares is $=r$; and the number of pairs of horizontal lines at a distance from each other of $n-s$ squares is $=s$. Hence the number of rectangles, breadth $n-r+1$ and depth $n-s+1$, [or say the number of $(n-r+1)(n-s+1)$ rectangles] is $=rs$.

For instance, $n=4$, the number of rectangles 44, 43, 34, &c., is shown in the diagram; hence the whole number of rectangles is $(1+2+3+4)^2 = 1^3+2^3+3^3+4^3$, and so for any value of n.

The same diagram shows that the whole number of squares is $= 1^2+2^2+3^2+4^2$; and so for any value of n.

	4	3	2	1
4	1	2		4
3	2	4	6	8
2	3	6	9	12
1	4	8	12	16

7982. (By Asparagus.)—If through any fixed point O on a given conicoid $u=0$ be drawn three chords OP, OQ, OR parallel to any three conjugate diameters of a second given conicoid $v=0$, the plane PQR will pass through a fixed point. [This includes as a particular case the theorem that, if OP, OQ, OR be any three chords at right angles, the plane PQR will pass through a fixed point, v being then a sphere and the fixed point lying on the normal to u at O. The present theorem is deducible from that by the method of projections, but is just as easily proved directly.]

Solution by Professor Wolstenholme, M.A., Sc.D.

Let $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy+px+qy+rz=0$ be the equation of one conicoid (u), and $\dfrac{(x-x_0)^2}{a^2}+\dfrac{(y-y_0)^2}{b^2}+\dfrac{(z-z_0)^2}{c^2}=1$ that of the second (v). Then, if (al_1, bm_1, cn_1), ( ), ( ) be direction cosines of three conjugate diameters of v, we shall have (l_1, m_1, n_1), &c. as direction cosines of a system of rectilinear axes. If OP, OQ, OR be chords of u, parallel to these conjugate diameters, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), coordinates of P, Q, R, then $\dfrac{x_1}{al_1}=\dfrac{y_1}{bm_1}=\dfrac{z_1}{cn_1}\ (\equiv k_1)$, whence we obtain

$$k_1(Aa^2l_1^2+Bb^2m_1^2+Cc^2n_1^2+2Fbc\,m_1n_1+\ldots)+(pal_1+qbm_1+rcn_1)=0,$$

Or, if $\lambda x+\mu y+\nu z=1$ be the equation of the plane PQR, then

$$k_1(\lambda al_1+\mu bm_1+\nu cn_1)=1,$$

whence, eliminating k_1, &c., there results

$$Aa^2 l_1^2 + Bb^2 m_1^2 + Cc^2 n_1^2 + 2Fbc\, m_1 n_1 + \dots + (pal_1 + qlm_1 + rcn_1)(\lambda al_1 + \mu lm_1 + \nu cn_1) = 0,$$

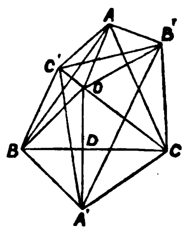
and, adding the two similar equations, we have

$$Aa^2 + Bb^2 + Cc^2 + p\lambda a^2 + q\mu b^2 + r\nu c^2 = 0,$$

which equation proves that the plane $\lambda x + \mu y + \nu z = 1$ passes through the fixed point $$\frac{x}{pa^2} = \frac{y}{qb^2} = \frac{z}{rc^2} = \frac{-1}{Aa^2 + Bb^2 + Cc^2}.$$

7756. (By the EDITOR.)—Construct a triangle whose angles are given, when there are also given the three distances from its vertices to a point in its plane.

Solutions by (1) Dr. CURTIS; (2) J. McDOWELL, M.A.



1. Take *any* triangle A′B′C′, equiangular with the one required, find within (or without) it a point O, such that OA′ : OB′ : OC′ shall be in the given ratios (the intersection of the loci of vertices of triangles having A′B′, B′C′ as bases and sides in known ratios), take on these lines the given lengths OA, OB, OC; ABC is plainly the required triangle.

2. Suppose ABC to be an equilateral triangle, and O a point within it, and that OA, OB, OC are given in magnitude. Draw OD perpendicular to BC, and produce it until DA′ = DO, &c. The angle A′CB′ = 120°, and A′C, B′C are known, therefore A′B′ is known. Hence we can construct the triangle A′B′C′; and, since C′A, B′A are given, therefore the point A is determined, &c.

[The problem is given in a generalised form as Quest. 8323, to which, indeed, Dr. CURTIS's solution also applies. The figure engraved refers more especially to Solution (2), a figure to (1) being unnecessary.]

8010. (By Professor KALIPRASANNA ROY, M.A.)—The perpendiculars from the angular points of an acute-angled triangle ABC on the opposite sides meet in P; and PA, PB, PC are taken for the sides of a new triangle; find (1) the condition that this should be possible; and, if it is, and the angles of the new triangle are α, β, γ, show that

$$(2)\quad 1 + \frac{\cos\alpha}{\cos A} + \frac{\cos\beta}{\cos B} + \frac{\cos\gamma}{\cos C} = \tfrac{1}{2}\sec A \sec B \sec C.$$

Solution by H. S. Jones; B. Hanumanta Rau, M.A.; *and others.*

PA, PB, PC are easily seen to be equal respectively to $2R\cos A$, $2R\cos B$, $2R\cos C$, where R is the circum-radius. The condition for the reality of the triangle is that any two of the quantities $\cos A$, $\cos B$, $\cos C$ should be greater than the third.

Again, $\cos\alpha = (\cos^2 B + \cos^2 C - \cos^2 A) / 2\cos B\cos C.$

Hence we have $$1 + \frac{\cos\alpha}{\cos A} + \frac{\cos\beta}{\cos B} + \frac{\cos\gamma}{\cos C}$$

$$= (2\cos A\cos B\cos C + \cos^2 A + \cos^2 B + \cos^2 C) / 2\cos A\cos B\cos C$$

$$= \tfrac{1}{2}\sec A\sec B\sec C,$$

for, when $A+B+C=0$, $2\cos A\cos B\cos C + \cos^2 A + \ldots + \ldots = 1$.

8326. (By Captain H. Brocard.)—On joint chaque sommet du triangle au centre de gravité G et à l'orthocentre H. Démontrer que les droites qui joignent les points de rencontre des précédentes avec la circonférence décrite sur GH comme diamètre passent par le centre K des symedianes.

Solution by R. F. Davis, M.A.

The symmedian centre K is the centroid of three masses proportional to a^2, b^2, c^2 placed at the angular points of the triangle (Quest. 8167); and this system can be resolved into the components, (i.) masses $\frac{1}{2}a^2$ at A, $\frac{1}{2}(a^2+b^2-c^2)$ at B, and $\frac{1}{2}(a^2+c^2-b^2)$ at C; (ii.) masses $\frac{1}{2}(b^2+c^2-a^2)$ each at A, B, C; (iii.) mass $a^2-\frac{1}{2}(b^2+c^2)$ at A. Let G′ be the projection of G upon the perpendicular AD, and H′ that of H upon the median AE. Then the system (i.) is equivalent to a mass $\frac{3}{2}a^2$ at G′; for

$$BD : CD :: a^2+c^2-b^2 : a^2+b^2-c^2;$$

(ii.) is equivalent to a mass $\frac{3}{2}(b^2+c^2-a^2)$ at G; while (ii.) and (iii.) are together equivalent to a mass $b^2+c^2-\frac{1}{2}a^2$ at H′, for

$$AH' : AG = 3AH.AD : 2AE^2 = 3(b^2+c^2-a^2) : 2b^2+2c^2-a^2.$$

Hence G′H′ passes through K.

7914. (By B. Reynolds, M.A.)—Prove that (1) the condition that the equation $x^4+qx^2+rx+s=0$ may have two coincident roots, is that $27r^4+4qr^2(q^2-36s)-16s(q^2-4s)^2=0$; and (2) find the condition for the same equation with px^3 added thereto.

Solutions by G. G. Storr, B.A.; R. Knowles, B.A.; *and others.*

Let $f(x) = x^4+px^3+qx^2+rx+s=0$, and for x write x/y, then

$$f(x, y) = x^4+px^3y+qx^2y^2+rxy^3+sy^4,\quad f'_x(x, y) = 4x^3+3px^2y+2qxy^2+ry^3,$$

$f'_y(x, y) = px^3 + 2qx^2y + 3rxy^2 + 4sy^3$; put $y = 1$, then the equations

$$4x^3 + 3px^2 + 2qx + r = 0 \quad \text{and} \quad px^3 + 2qx^2 + 3rx + 4s = 0 \quad \text{.........(1, 2)}$$

have one root the same in each. Eliminating the first and last terms from these equations, there results

$$(16s - pr)\, x^2 + 2\,(6ps - qr)\, x + 8qs - 3r^2 = 0,$$
$$(3p^2 - 8q)\, x^2 + 2\,(pq - 6r)\, x + rp - 16s = 0,$$

which will also have one identical root. Writing these last for brevity $ax^2 + bx + c = 0$, and $a'x^2 + b'x + c' = 0$, the condition is either

$$(a'c - ac')^2 = (a'b - ab')\,(b'c - bc'), \quad \text{or} \quad (ac' - a'c)^2 = (ab' - a'b)\,(bc' - b'c).$$

Putting $p = 0$, we have the result in the question.

7977. (By W. J. McClelland, B.A.)—Given the three perpendiculars of a spherical triangle; find the sides.

Solution by the Proposer.

Since $\sin a = \dfrac{2n}{\sin p}$, ..., it follows that

$$\frac{1}{\sin p} = \frac{\sin a}{2n}, \quad \frac{1}{\sin q} = \frac{\sin b}{2n}, \quad \frac{1}{\sin r} = \frac{\sin c}{2n} \quad \text{..............(1).}$$

It remains therefore to calculate the function n. By equation (1), we have

$$\left(\frac{1}{\sin p} + \frac{1}{\sin q} + \frac{1}{\sin r}\right)^2 = \left(\frac{\sin a + \sin b + \sin c}{2n}\right)^2,$$

or

$$\left(\frac{1}{\sin p} + \frac{1}{\sin q} + \frac{1}{\sin r}\right)^2 - 1$$
$$= \frac{2\,(1 + \sin b \sin c + \sin c \sin a + \sin a \sin b - \cos a \cos b \cos c)}{4n^2}.$$

But, from a well-known theorem in Plane Trigonometry,

$$\{\sin s + \sin (s-a) + \sin (s-b) + \sin (s-c)\}^2$$
$$= 2\,(1 + \sin b \sin c + \sin c \sin a + \sin a \sin b - \cos a \cos b \cos c),$$

Hence we get

$$\left(\frac{1}{\sin p} + \frac{1}{\sin q} + \frac{1}{\sin r}\right)^2 - 1 = \left(\frac{\sin s + \sin (s-a) + \sin (s-b) + \sin (s-c)}{2n}\right)^2 \quad \text{............(2).}$$

By changing the sign of $\sin p$ on the left-hand side of equation (2), and of $\sin a$ on the right-hand side, which we may do by equations (1), it likewise follows, as is otherwise evident by symmetry, that

$$\left(-\frac{1}{\sin p} + \frac{1}{\sin q} + \frac{1}{\sin r}\right)^2 - 1 = \left(\frac{\sin s + \sin (s-a) - \sin (s-b) - \sin (s-c)}{2n}\right)^2.$$

Now, for the sake of brevity, let

$$[(\operatorname{cosec} p+\operatorname{cosec} q+\operatorname{cosec} r)^2-1]^{\frac{1}{2}} = \mathrm{P},$$
$$[(-\operatorname{cosec} p+\operatorname{cosec} q+\operatorname{cosec} r)^2-1]^{\frac{1}{2}} = \mathrm{Q},$$
$$[(\operatorname{cosec} p-\operatorname{cosec} q+\operatorname{cosec} r)^2-1]^{\frac{1}{2}} = \mathrm{R};$$
$$[(\operatorname{cosec} p+\operatorname{cosec} q-\operatorname{cosec} r)^2-1]^{\frac{1}{2}} = \mathrm{S};$$

Then
$$\sin s+\sin(s-a)+\sin(s-b)+\sin(s-c) = 2n\mathrm{P},$$
$$\sin s+\sin(s-a)-\sin(s-b)-\sin(s-c) = 2n\mathrm{Q},$$
$$\sin s-\sin(s-a)+\sin(s-b)-\sin(s-c) = 2n\mathrm{R},$$
$$\sin s-\sin(s-a)-\sin(s-b)+\sin(s-c) = 2n\mathrm{S}.$$

From these equations we have at once

$$\left.\begin{array}{ll} 2\sin s = n(\mathrm{P+Q+R+S}), & 2\sin(s-a) = n(\mathrm{P+Q-R-S}) \\ 2\sin(s-b) = n(\mathrm{P-Q+R-S}), & 2\sin(s-c) = n(\mathrm{P-Q-R+S}) \end{array}\right\} \ldots(3).$$

Multiplying these equations (3) together, remembering that

$$n^2 = \sin s\sin(s-a)\sin(s-b)\sin(s-c),$$

we have, finally,

$$n = \frac{4}{[(\mathrm{P+Q+R+S})(\mathrm{P+Q-R-S})(\mathrm{P-Q+R-S})(\mathrm{P-Q-R+S})]^{\frac{1}{2}}}.$$

7621. (By Asutosh Mukhopâdhyây, B.A.)—If δ_1 represents the distance of a point P in the plane of a given triangle (area = Δ_2) from the centre of the circumscribing circle (diameter = D), and Δ_1 the area of the triangle formed by joining the feet of the perpendiculars on the sides from P; show that (1) $\dfrac{\Delta_1}{\Delta_2} = \pm\dfrac{\mathrm{D}^2-4\delta_1^2}{4\mathrm{D}^2}$; and hence (2) if α, β, γ denote the distance of P from A, B, C, and Δ_3, the area of the triangle whose sides are $\alpha\sin\mathrm{A}$, $\beta\sin\mathrm{B}$, $\gamma\sin\mathrm{C}$, then

$$2a^2\operatorname{cosec}\mathrm{A}\sin\mathrm{B}\sin\mathrm{C} = \alpha^2\sin 2\mathrm{A}+\beta^2\sin 2\mathrm{B}+\gamma^2\sin 2\mathrm{C}\pm 8\Delta_3;$$

(3) if perpendiculars be drawn from the angular points of a triangle upon the three sides, and from the extremities of any one of these perpendiculars lines be drawn at right angles to the other two perpendiculars and the other two sides, the extremities of these four perpendiculars lie in a line parallel to the line joining the extremity of the other two perpendiculars drawn from the angular points on the sides.

Solution by B. Hanumanta Rau, B.A.

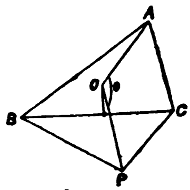
1. Let O be the centre of the circumscribed circle, and let PD, PE, PF be the perpendiculars from P on the sides BC, CA, AB.

Since Δ_1 depends obviously on the distance δ_1 only, suppose P to be in the line AO produced, then $\mathrm{PD} = \delta_1\cos(\mathrm{C-B})-\mathrm{R}\cos\mathrm{A}$,

$$\mathrm{PE} = (\delta_1+\mathrm{R})\cos\mathrm{B}, \text{ and } \mathrm{PF} = (\delta_1+\mathrm{R})\cos\mathrm{C},$$
$$\Delta_1 = \tfrac{1}{2}(\mathrm{PD}\,.\,\mathrm{PE}\sin\mathrm{C}+\mathrm{PD}\,.\,\mathrm{PF}\sin\mathrm{B}-\mathrm{PE}\,.\,\mathrm{PF}\sin\mathrm{A})$$
$$= \tfrac{1}{2}(\delta_1^2-\mathrm{R}^2)\sin\mathrm{A}\sin\mathrm{B}\sin\mathrm{C}.$$
$$\Delta_2 = \frac{a^2\sin\mathrm{B}\sin\mathrm{C}}{2\sin\mathrm{A}} = 2\mathrm{R}^2\sin\mathrm{A}\sin\mathrm{B}\sin\mathrm{C},$$

therefore $$\frac{\Delta_1}{\Delta_2} = \frac{\delta_1^2 - R^2}{4R^2} = \frac{4\delta_1^2 - D^2}{4D^2}.$$

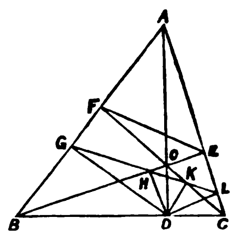
The double sign $\pm$ must be taken, since D might be greater than $2\delta_1$.



2 $$\alpha^2 = \delta_1^2 + R^2 - 2\delta_1 R\cos\theta,$$
$$\beta^2 = \delta_1^2 + R^2 - 2\delta_1 R\cos(2C+\theta),$$
$$\gamma^2 = \delta_1^2 + R^2 - 2\delta_1 R\cos(\theta - 2B),$$
$$16\Delta_3^2 = 4\beta^2\gamma^2\sin^2 B\sin^2 C - (\beta^2\sin^2 B + \gamma^2\sin^2 C - \alpha^2\sin^2 A)^2$$
$$= 4\beta^2\gamma^2\sin^2 B\sin^2 C - 4\sin^2 B\sin^2 C\left\{(\delta_1^2 + R^2)\cos A - 2\delta_1 R\cos(\theta + C - B)\right\}^2;$$

therefore $$4\Delta_3^2 = \sin^2 A\sin^2 B\sin^2 C\left\{(\delta_1^2+R^2)^2 - 4\delta_1^2 R^2\right\};$$
$$\pm 2\Delta_3 = \sin A\sin B\sin C(\delta_1^2 - R^2),$$
$$\alpha^2\sin 2A + \beta^2\sin 2B + \gamma^2\sin 2C = (\delta_1^2 + R^2)\,4\sin A\sin B\sin C;$$

therefore $$\alpha^2\sin 2A + \beta^2\sin 2B + \gamma^2\sin 2C \mp 8\Delta_3 = 2R^2 . 4\sin A\sin B\sin C = 2a^2\operatorname{cosec} A\sin B\sin C.$$



3. Draw DG and DH perpendicular to AB and BE, join GH, and produce to meet CF and CE at K and L. A circle passes through BGHD, and another through BFEC,

therefore $\angle DGH = \angle DBH = \angle EBC = \angle EFC$,

and DG, CF are parallel; therefore GH is parallel to FE.

Again, because $\angle HOD = \angle OBD = \angle HGD = \angle OKH$; therefore a circle passes through OKDH. But $\angle OHD$ = a right angle; therefore $\angle OKD$ = a right angle.

Again, $\angle HKD = \angle HOD$ = complement of $\angle OBC = \angle DCL$,

therefore $\angle DKL + \angle DCL$ = two right angles.

A circle passes through DKLC, therefore $\angle DLC = \angle DKC$ = a right angle. Hence the feet of the perpendiculars DG, DH, DK, and DL from D on the lines BA, BE, CE and CA are in one straight line GHKL, which is parallel to FE.

7911. (By B. Hanumanta Rau, B.A.)—If A pays £760 to B in order to receive from B a monthly payment of £100 for ten months, the first being received a month after he made the payment to B, find the rate of interest A has charged.

Solution by D. Biddle.

Assuming that $760 + 7\cdot60x$ would be an equivalent amount, payable in a year from date, $100 + \frac{11}{12}x$ will be the value of the first instalment, $100 + \frac{10}{12}x$ of the second, and so on down to $100 + \frac{2}{12}x$, the value of the last. Summing these, we obtain the equation $1000 + 5\cdot41\dot{6}x = 760 + 7\cdot60x$,

whence x = £109. 18*s*. 6*d*. per cent. nearly, equivalent to the payment at the year's end of £1,595. 8*s*. 5*d*. principal and interest.

But, if we assume that the rate of interest is in no way affected by the date at which the principal is returnable, then we may regard each monthly instalment as the payment back of a portion of the principal, with interest on that portion from date of loan. Let x = rate of interest per cent. per annum, and y = rate of interest per £1 per month, then $x = 1200y$, and the sum of the several instalments of principal may be rendered as follows: $\frac{100}{1+y}+\frac{100}{1+2y}+\ldots+\frac{100}{1+10y}=760$,

whence $y = \cdot 062$ nearly, and $x = 74\frac{1}{3}$ per cent. per annum.

But, that the benefit to the usurer is by no means over-stated in the former of the two solutions, will appear when we take into consideration the enormous increase in capital which would accrue to him at the end of the ten months, supposing he could make, on a given day, a similar arrangement with eight different persons, and had a succession of applicants month by month. At the end of the first month, he would have more than enough to lend the same sum to a ninth client; at the end of the second month, he could oblige a tenth; afterwards two, and even three at once; so that by the time the original eight debtors had paid up, he would possess in unexpired loans an amount equal to £12,600, which would bring him in £1,900 the very next month.

It may further be remarked that great laxity is allowed in the calculation of rates of interest. For instance, a person borrows £2,000 at 5 per cent. per annum, and is expected to pay as interest £50 half-yearly, or perhaps £25 quarterly, the former arrangement raising the rate of interest to £5. 1*s*. 3*d*., and the latter to £5. 1*s*. 10½*d*.

Note on Question 7415. *By* Dr. Curtis.

The analytical discussion of this question (given on p. 97 of Vol. xlii.) contains implicitly the complete solution of the problem; but the proposer takes exception to it *as not including the second chord of intersection*, although in his statement of the problem he speaks of the enveloping conic of *the* common *chord*. The fact is, an ellipse is *fully* defined by the equation $r = eP \equiv e\,(x\cos\alpha + y\sin\alpha - p)$, where e is essentially positive, as for every point on the curve r and P retain the same sign (taken to be positive), as neither passes through infinity or zero. In fact, two ellipses which have one focus in common can only have *one* common chord. In the case of the hyperbola, we must regard e as affected by the ambiguous sign $\pm$. If then one or other of the conics be a hyperbola, or e *both*, or e', or *both*, must be regarded as including the ambiguous sign; and, *thus understood*, the equations at the head of page 97, Vol. 42, give two chords of intersection, and two envelopes, one defined absolutely by the equation as given in the solution referred to, and the other obtained from it by changing the sign of *one* of the quantities e and e', as changing the signs of *both* leaves the equation unaltered.

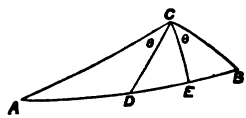
[Mr. Simmons writes, that, "while admitting the justice of Dr. Curtis's criticism as to the over-brevity of the statement of Question 7415, he is fortunate in having called forth the above able and ingenious completion of a solution which he always considered superior to his own. It will now

be seen that p. 97 of Vol. XLII. contains two independent solutions, analytical and geometrical, of the following extensions of Question 4417 (given on p. 117 of Vol. XXXIX.) :—

"'If two conics A and B have a common focus, about which one of them revolves, then (1) the envelope of either of those common chords which initially are perpendicular to the common axis is a conic; (2) either of which conics is completely determined when the positions of the nearer directrices (to the common focus), and the *ratio* of the eccentricities, of A and B are given; (3) both of which conics pass through two points depending only on the positions of the aforesaid directrices, *and entirely independent of the eccentricities*, of A and B; (4) these two common chords and two directrices are concurrent throughout the revolution, the locus of their intersection being, of course, the stationary directrix; (5) the pencil formed by them is always harmonic.'"]

8029. (By W. J. McClelland, B.A.)—If s denote a symmedian of a spherical triangle, drawn from the vertex C to the opposite side c; show that (1) $\tan s = (\sin^2 a + \sin^2 b + 2 \sin a \sin b \cos C)^{\frac{1}{2}} / (\cot a \sin b + \cot b \sin a)$, and (2) the analogous expression *in plano* $s = ab\,(2a^2 + 2b^2 - c^2)^{\frac{1}{2}} / (a^2 + b^2)$.

Solution by B. Hanumanta Rau, B.A.; G. G. Storr, B.A.; *and others.*



Let CD be the median, and CE the symmedian. From $\triangle$CAD,

$$\cot\tfrac{1}{2}c \sin b = \cos b \cos A + \cot\theta \,.\, \sin A,$$

$$\therefore\ \cot\theta = \left\{(1+\cos c)\sin b - \sin c \cos b \cos A\right\} / \sin c \,.\, \sin A$$
$$= (\sin b + \sin a \,.\, \cos C) / \sin a \,.\, \sin C;$$

therefore $\quad \operatorname{cosec}\theta = (\sin^2 a + \sin^2 b + 2\sin a \sin b \cos C)^{\frac{1}{2}} / \sin a \sin C.$

Again, from the $\triangle$CEB, $\cot s \,.\, \sin a = \cot B \sin\theta + \cos a \,.\, \cos\theta$;

therefore $\tan s = \sin a \,.\, \operatorname{cosec}\theta / (\cot B + \cos a \,.\, \cot\theta)$

$$= (\sin^2 a + \sin^2 b + 2\sin a \sin b \cos C)^{\frac{1}{2}} / (\cot a \sin b + \cot b \sin a).$$

For $\cot B + \cos a \,.\, \cot\theta = (\cos b - \cos a \cos c + \cos a \,.\, \sin^2 b + \cos a\,(\cos c - \cos a \cos b) / \sin a \sin b \sin C,$

$$= (\cot a \sin b + \cot b \sin a) / \sin C$$

Hence, *in plano*, $\dfrac{s}{r} = \left(\dfrac{a^2}{r^2} + \dfrac{b^2}{r^2} + 2\,\dfrac{ab}{r^2}\cos C\right)^{\frac{1}{2}} \Big/ \left(\dfrac{b}{a} + \dfrac{a}{b}\right) =$ the result.

7909. (By Professor Byomakesa Chakravarti, M.A.) — If three different persons have each to name an integer not greater than n, find the chance that the integers named will be such that every two are together greater than the third.

Solutions by Rev. T. C. Simmons, M.A.; Prof. Nash, M.A.; *and others.*

Let A, B, C be the persons; a, b, c the integers named by them. Then, excluding zero numbers, and considering the cases in which a is the

greatest of the three integers, the number of ways in which $b+c$ cannot exceed a is the sum of the coefficients as far as that of x^a in the expansion $(x+x^2+x^3+\ldots+x^n)^2$, that is, remembering that $a \not> n$,

$$1+2+3+\ldots+(a-1), \quad \text{or} \quad \tfrac{1}{2}a(a-1);$$

hence the chance that $b+c$ is not greater than a is

$$\frac{1}{2n^3}\left\{2(2-1)+3(3-1)\ldots+n(n-1)\right\} \quad \text{or} \quad \frac{n^2-1}{6n^2},$$

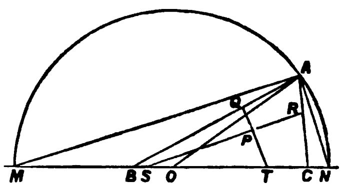
which is also the value of each of the corresponding chances when b or c is the greatest integer.

Therefore the whole chance that the sum of the two least integers is not greater than the third is $n^2-1 / 2n^2$, and the chance that it *is* greater is $n^2+1 / 2n^2$, the required result.

Putting $n=\infty$, we see that, when there are no limitations of magnitude, the chance that of three random integers any two are greater than the third is $\frac{1}{2}$. Hence, taking unity to denote a line or element of infinitesimal length, we arrive at this result, proposed for independent solution as Question 8035:—"Three lines being chosen at random, it is an even chance that they can form the sides of a triangle." [The above solution of the Question is simpler than that given on p. 91 of Vol. IX.]

3869. (By the Editor.)—Divide into four equal parts, by two straight lines perpendicular to each other, (1) a triangle, (2) a quadrangle.

Solutions by (1) D. Biddle; (2) *the* Proposer.



1. The lines will cut one of the sides in two points. Let BC, in the triangle ABC, be the side so cut, in S, T; and let BC = unity. Suppose AM, AN to be drawn parallel to the required lines. Then we have $2CR \cdot CS = AC$; $2BQ \cdot BT = AB$; $CM = 2CS^2$; $BN = 2BT^2$; $\triangle PST : \triangle ABC = 1 : 4$; $\triangle ABC : \triangle AMN = 1 : MN$; and $ST^2 : MN^2 = \triangle PST : \triangle AMN$; whence $MN = 4ST^2 = 2CS^2 + 2BT^2 - 1$. But $ST = CS + BT - 1$; therefore $CS = \frac{1}{2}[1+ST+(3ST^2-2ST)^{\frac{1}{2}}]$, and $BT = \frac{1}{2}[1+ST-(3ST^2-2ST)^{\frac{1}{2}}]$. Consequently $\frac{1}{2}(3ST^2-2ST)^{\frac{1}{2}}$ = the distance from the mid-point of BC to the mid-point of ST. The two coincide when $3ST^2 = 2ST$, *i.e.*, when $ST = \frac{2}{3}$. But, when one end of ST coincides with B or C, then $ST = \frac{1}{2}\sqrt{2}$; this occurs when AB = BC, or AC = BC. These, accordingly, are the limits of ST; it varies in length from $\frac{2}{3}$ when its mid-point coincides with the centre of BC, to $\frac{1}{2}\sqrt{2}$ when its mid-point is $\frac{1}{2}(1-\frac{1}{2}\sqrt{2})$ on one or other side of that centre.

Now, if we suppose MN bisected in O, and a semi-circle drawn on it, the apex A may be anywhere on the circumference, whilst the points S, T remain unchanged. But P will under these circumstances vary with A, the radius ($=\frac{1}{2}$ST), from the mid-point of ST to P, keeping parallel to OA. For every position of O and distance OA (varying from $\frac{3}{5}$ to 1), there is thus a distinct series of triangles, differing greatly in other respects, but having S and T identically placed. And what we have to find, in any given triangle, is the line OA, that shall so divide it as to make $BO + OA = 2BT^2$, and $CO + OA = 2CS^2$, whilst $OA = 2(BT + CS - 1)^2$.

Let $OA = x$, and $ST = y$, then $x = (\frac{1}{2}y)^{\frac{1}{2}}$, and we have

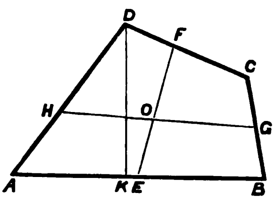
$$2CS^2 = 2y^2 + (4y^4 - AB^2 \sin^2 B)^{\frac{1}{2}} + AC\cos C = \tfrac{1}{2}\left\{1 + y + (3y^2 - 2y)^{\frac{1}{2}}\right\}^2;$$

therefore $\quad (4y^4 - AB^2\sin^2 B)^{\frac{1}{2}} - (1+y)(3y^2 - 2y)^{\frac{1}{2}} = \frac{1}{4} - AC\cos C,$

whence

$$\begin{aligned} & y^8 - 8y^7 + 18y^6 - 4y^5 - \left\{15 + 14\left(AC^2 + \tfrac{1}{4} - AC\cos C\right)\right\} y^4 \\ & + \left\{4 + 16AB^2\sin^2 B - 8\left(AC^2 + \tfrac{1}{4} - AC\cos C\right)\right\} y^3 \\ & + \left\{4 + 2\left(AC^2 + \tfrac{1}{4} - AC\cos C\right) - 4AB\sin^2 B\right\} y^2 \\ & + \left\{4\left(AC^2 + \tfrac{1}{4} - AC\cos C\right) - 8AB^2\sin^2 B\right\} y \\ & + \left(AC^2 + \tfrac{1}{4} - AC\cos C\right)^2 = 0. \end{aligned}$$

From this equation y can be found, and thence x, and the required lines can readily be constructed.



2. In the annexed diagram, where EF, GH are the quadrisecting lines, and DK is perpendicular to AB, put $AB = l$, $AK = m$, $DK = n$, area of $ABCD = s$; and with A as origin, and AB as axis of x, let the rectangular equations of BC, CD, DA, EF, GH be $x = a_1 y + b_1$, $y = a_2 x + b_1$, $x = cy$, $y = ux + v$, $x = -uy + w$; then the coordinates of E, F, G, H, O will be $(-v/u, 0)$,

$$\left(\frac{v - b_2}{a_2 - u}, \frac{a_2 v - b_2 u}{a_2 - u}\right), \left(\frac{a_1 w + b_1 u}{a_1 + u}, \frac{w - b_1}{a_1 + u}\right), \left(\frac{cw}{c+u}, \frac{w}{c+u}\right),$$

$$\left(\frac{w - uv}{u^2 + 1}, \frac{v + uw}{u^2 + 1}\right);$$

and the areas of AEFD, ABGH, AEOH being found herefrom, we have

$$\frac{u(v - b_2)}{a_2 - u} - \frac{a_2 v - b_2 u}{a_2 - u}\left(\frac{v}{u} + m\right) = s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{a_1 w + b_1 u}{a_1 + u}\left(\frac{w}{c+u}\right) + \frac{w - b_1}{a_1 + u}\left(l - \frac{cw}{c+u}\right) = s \quad \ldots\ldots\ldots\ldots\ldots(2),$$

$$\frac{w - uv}{u^2 + 1}\left(\frac{w}{c+u}\right) - \frac{v + uw}{u^2 + 1}\left(\frac{v}{u} + \frac{cw}{c+u}\right) = \tfrac{1}{2}s \ldots\ldots\ldots\ldots\ldots(3).$$

The equations will give the values of u, v, w, which determine the positions of the quadrisecting lines.

If we put $v = pu$, $w = q(c+u)$, $L_1 = a_2 u$, $M_1 = a_2 mu - b_1 u - nu$, $N_1 = a_2 s + b_2 u - su - b_1 mu$, $L_2 = cu - a_1 u - a_1 c + c^2$, $M_2 = b_1 c - bu_1 - cl - lu$, $N_2 = a_1 s + su + b_1 l$, $L_3 = 2u$, $M_3 = 4u^2 + 4cu$, $N_3 = 2c^2 u + 2cu^2 - 2c - 2u$,

$Q_3 = -s - su^2$. Equations (1), (2), (3), become

$$L_1 p^2 + M_1 p + N_1 = 0, \quad L_2 q^2 + M_2 q + N_2 = 0 \ldots\ldots\ldots\ldots(4, 5),$$

$$L_3 p^2 + M_3 pq + N_3 q^2 = Q_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6);$$

and, since L, M, N, Q are functions of u only, the unknowns in (4), (5), (6) are p, q, u. Thus, by substituting in (6) the values of p, q derived from (4), (5), we obtain an equation in u, from which u, and thence p, q, v, w can all be found.

The working-out of these equations, as set forth above, would furnish a solution of the problem in part (2) of the question, and, by treating one of the sides of the quadrilateral as zero, a solution of part (1) might be derived therefrom.

[The problem in (2), for quadrisecting a *quadrangle*, was originally proposed by OZANAM, as an unsolved problem in his *Mathematical Recreations* (see Vol. I., p. 357, Prob. 19, in HUTTON'S Translation).]

7948. (By ASÛTOSH MUKHOPÂDHYÂY.)—Tangents are drawn to any central conic, so that the squares of the intercepts on the minor axis are in arithmetical progression; show that the squares of the sines of the angles which the tangents make with the minor axis are in harmonic progression.

Solution by Rev. T. GALLIERS, M.A.

Let θ_1, θ_2, θ_3 be the inclinations of three successive tangents of the series to the minor axis, and γ_1, γ_2, γ_3 the intercepts of the tangents on the minor axis. The equation of the first tangent will be

$$y \sin\theta_1 - x\cos\theta_1 = (a^2\cos^2\theta_1 \pm b^2\sin^2\theta_1)^{\frac{1}{2}},$$

and, putting $x = 0$, &c., we have, (+ for ellipse or circle, − for hyperbola),

$$\gamma_1^2 = a^2\cot^2\theta_1 \pm b^2, \quad \gamma_2^2 = a^2\cot^2\theta_2 \pm b^2, \quad \gamma_3^2 = a^2\cot^2\theta_3 \pm b^2.$$

By the question $\quad 2\gamma_2^2 = \gamma_1^2 + \gamma_3^2, \quad$ therefore $2\cot^2\theta_2 = \cot^2\theta_1 + \cot^2\theta_3$,

therefore $\qquad 2(1+\cot^2\theta_2) = (1+\cot^2\theta_1) + (1+\cot^2\theta_3)$,

or $\qquad 2\,\mathrm{cosec}^2\theta_2 = \mathrm{cosec}^2\theta_1 + \mathrm{cosec}^2\theta_3$,

therefore $\sin^2\theta_1$, $\sin^2\theta_2$, $\sin^2\theta_3$ are in H. P., and the same thing may be proved true for any three consecutive tangents of the series.

7174. (By W. J. C. SHARP, M.A.)—The result of translating a system of vectors β along a vector α, and rotating them through an angle twice the angle of a quaternion q about its axis, is

$$q(\alpha+\beta)q^{-1} \text{ or } \alpha + q\beta q^{-1},$$

according to the order in which the operations are performed. Show that the expressions are identical if α be the axis of q. (This is one of the advantages of Dr. BELL's theory of screws.)

Solution by the PROPOSER.

$$q(\alpha+\beta)q^{-1} = q\alpha q^{-1} + q\beta q^{-1},$$

since quaternion multiplication is distributive, and

$$q\alpha q^{-1} = \frac{1}{T^2q}\,q\alpha\,Kq = \frac{1}{T^2q}(Sq+Vq)\,\alpha\,(Sq-Vq)$$

$$= \frac{1}{S^2q-V^2q}\left\{S^2q.\alpha + Sq.Vq.\alpha - Sq.\alpha Vq - Vq.\alpha.Vq\right\}$$

$$= \alpha + \frac{1}{T^2q}\left\{V^2q\alpha + Sq.Vq.\alpha - Sq.\alpha Vq - Vq.\alpha.Vq\right\}$$

$$= \alpha + \frac{1}{T^2q}\left\{qVq.\alpha - q\alpha Vq\right\}$$

$$= \alpha + \frac{4}{T^2q}\,qV(Vq.\alpha) = \alpha \text{ if } V(Vq.\alpha) = 0;$$

i.e., if α be parallel to Vq.

NOTE ON QUEST. 8209 (see Vol. XLIV., p. 86). *By* W. J. MCCLELLAND, B.A.

1. Generally, if a triangle XYZ be inscribed in a triangle ABC, such that the sides BC and YZ, CA and ZX, AB and XY have the same inclination α in the same aspect; then (1) the triangles are similar; (2) they have a Brocard-point ω in common, viz., the centre of similitude of the triangles; (3) the ratio of similitude is $\sin(\theta+\alpha) : \sin\theta$, where θ is the Brocard-angle. The last property may be thus proved *geometrically*:— Join the common Brocard-point to the corners of the triangles. Since these lines divide the triangles into three pairs of similar triangles, therefore $\dfrac{\Delta.ABC}{\Delta.XYZ} = \dfrac{\Delta.C\omega A}{\Delta.Y\omega Z} = \dfrac{\omega C^2}{\omega Y^2} = \dfrac{\sin^2(\theta+\alpha)}{\sin^2\theta}$, therefore, &c.

Thus $R' : R = \sin\theta : \sin(\theta+\alpha)$. Now, if $\alpha = 0°$, $R' = R$, and the circumcircles coincide.

2. Again, if $\theta+\alpha = \pi-\theta$, $R = R'$; hence, *if the inclination of the sides of XYZ is equal to twice the complement of the Brocard-angle, the circumcircles of XYZ and ABC are equal.*

3. Let $\alpha = \theta$. Then $R' : R = \sin\theta : \sin 2\theta$, or $R = 2R'\cos\theta$; an expression for the radius R' of the "T. R." circle in terms of the radius of the circumcircle and Brocard-angle.

4. On substituting the value of $\cos\theta$ in terms of the sides, it follows at once, remembering that $\tan\theta = \dfrac{4\Delta}{a^2+b^2+c^2}$, that $\dfrac{R'^2}{R^2} = \dfrac{b^2c^2+c^2a^2+a^2b^2}{(a^2+b^2+c^2)^2}$. (See CASEY's *Sequel to Euclid*, p. 169.)

5. The minimum value of R′ from the equation $\dfrac{R'}{R} = \dfrac{\sin\theta}{\sin(\alpha+\theta)}$, is obtained by making $\alpha+\theta = 90°$, in which case the angle α of inclination is the complement of the Brocard-angle, and the value of the minimum diameter of the circumcircle of XYZ is found to be $abc \,/\, (b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}$.

7860. (By ASÛTOSH MUKHOPÂDHYÂY, B.A., F.R.A.S.)—Prove that

$$\begin{aligned} P \equiv\ & (x-5y-2z)^3-(2x-y-5z)^3-(5x+2y-z)^3 \\ & +3(x-5y-2z)(4x-5y-3z)(5x+2y-z) \\ & +3(2x-y+5z)(5x+3y-4z)(5x+2y-z) \\ & -3(x-5y-2z)(3x-4y+5z)(2x-y+5z) \\ & -6(x-5y-2z)(2x-y+5z)(5x+2y-z) = 2^3 . 3^5 . xyz. \end{aligned}$$

Solution by the PROPOSER; G. G. STORR, B.A.; *and others*.

Put $x-5y-2z = -\alpha,\quad 2x-y+5z = \beta,\quad 5x+2y-z = \gamma$ ....(1);

then, assuming $4x-5y-3z = -\lambda\alpha+\mu\gamma$,

and equating coefficients, we have

$$\lambda+5\mu = 4,\quad -5\lambda+2\mu = -5,\quad -2\lambda-\mu = -3,$$

which system is seen to be consistent, viz., $\lambda = \frac{11}{9}$, $\mu = \frac{5}{9}$, therefore

$$4x-5y-3z = \tfrac{1}{9}(5\gamma-11\alpha);$$

similarly $5x+3y-4z = \frac{1}{9}(11\gamma-5\beta)$, $3x-4y+5z = \frac{1}{9}(11\beta-5\alpha)$;

hence
$$\begin{aligned} 3P &= 18\alpha\beta\gamma+(11\alpha-5\gamma)\alpha\gamma+(11\gamma-5\beta)\gamma\beta \\ &\qquad +(11\beta-5\alpha)\alpha\beta-3(\alpha^3+\beta^3+\gamma^3) \\ &= (-\alpha+\beta+3\gamma)(3\alpha-\beta+\gamma)(\alpha+3\beta-\gamma) \\ &= 18x . 18y . 18z, \text{ from (1)} = 2^3 . 3^6 . xyz. \end{aligned}$$

[When $x = 0$ we find that P reduces to zero, hence x is a factor of P; and, as P is symmetrical, y and z must also be factors, hence xyz is a factor, that is to say, $P = Mxyz$; but P is of three dimensions, therefore M is a number, and by putting $x=y=z=1$, we readily find $M = 2^3 . 3^5$.]

8105. (By W. J. GREENSTREET, B.A.)—If O be the mid-point of an equilateral spherical triangle ABC, and P any point on the surface of the sphere, such that $m\sec PA = n\sec PB = p\sec PC$, prove that

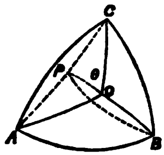
$$(\tan PO \tan AO)^2 = 4(m^2+n^2+p^2-pn-nm-mp) \,/\, (m+n+p)^2.$$

Solution by W. J. McClelland, M.A.

$$\cos^2 PA + \cos^2 PB + \cos^2 PC - \cos PB \cos PC - \cos PC \cos PA - \cos PA \cos PB$$
$$= (\cos PA + \omega \cos PB + \omega^2 \cos PC)(\cos PA + \omega^2 \cos PB + \omega \cos PC) \ldots\ldots (1).$$

Now, in the figure let $\angle POC = \theta$,

also let $AO = BO = CO = R$ and $PO = \delta$.



Then

$$\cos PA = \cos R \cos \delta + \sin R \sin \delta \cos(\tfrac{2}{3}\pi - \theta) \ldots\ldots (2),$$
$$\cos PB = \cos R \cos \delta + \sin R \sin \delta \cos(\tfrac{2}{3}\pi + \theta) \ldots\ldots (3),$$
$$\cos PC = \cos R \cos \delta + \sin R \sin \delta \cos \theta \ldots\ldots\ldots\ldots (4).$$

Hence by addition

$$\cos PA + \cos PB + \cos PC = 3 \cos R \cos \delta \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Multiplying (3) by ω, and (4) by ω^2, and adding,

$$\cos PA + \omega \cos PB + \omega^2 \cos PC$$
$$= \sin R \sin \delta \left\{\cos(\tfrac{2}{3}\pi - \theta) + \omega \cos(\tfrac{2}{3}\pi + \theta) + \omega^2 \cos \theta\right\}.$$

Similarly

$$\cos PA + \omega^2 \cos PB + \omega \cos PC$$
$$= \sin R \sin \delta \left\{\cos(\tfrac{2}{3}\pi - \theta) + \omega^2 \cos(\tfrac{2}{3}\pi + \theta) + \omega \cos \theta\right\}.$$

The product of the latter equations gives on the right-hand side $\frac{9}{4} \sin^2 R \sin^2 \delta$, which by the aid of (5) becomes $\frac{1}{4} \tan^2 R \tan^2 \delta$; therefore, &c.

[It is worth noticing that $\left\{\cos(\tfrac{2}{3}\pi - \theta) + \omega \cos(\tfrac{2}{3}\pi + \theta) + \omega^2 \cos \theta\right\}$

$$\times \left\{\cos(\tfrac{2}{3}\pi - \theta) + \omega^2 \cos(\tfrac{2}{3}\pi + \theta) + \omega \cos \theta\right\}$$

$$= \begin{vmatrix} 1 & \cos\theta & \cos(\tfrac{2}{3}\pi + \theta) \\ 1 & \cos(\tfrac{2}{3}\pi - \theta) & \cos\theta \\ 1 & \cos(\tfrac{2}{3}\pi + \theta) & \cos(\tfrac{2}{3}\pi - \theta) \end{vmatrix} = 3\left\{\cos^2(\tfrac{2}{3}\pi - \theta) - \cos\theta \times \cos(\tfrac{2}{3}\pi + \theta)\right\} = \tfrac{9}{4}\Big].$$

8134. (By R. Tucker, M.A.)—If DEF is the pedal triangle of ABC (AD, BE, CF); H is mid-point of perpendicular EG on DF: prove that $\tan CDH = \tan^3 A$.

Solution by Asûtosh Mukhopâdhyây, M.A.; Rev. T. Galliers, M.A.; *and others.*

Let $\angle CDH = \theta$, then $\angle GDH = 180° - A - \theta$, also $\angle GDE = 180° - 2A$; and, by the question, $\tan GDE : \tan GDH = GE : GH = 2 : 1$,

that is, $$\tan 2A = 2\tan(A + \theta),$$

or $$2\tan A / (1 - \tan^2 A) = 2(\tan A + \tan\theta) / (1 - \tan A \tan\theta),$$

therefore $$\tan A(1 - \tan A \tan\theta) = (1 - \tan^2 A)(\tan A + \tan\theta)$$

$$\tan A - \tan^2 A \tan\theta = \tan A - \tan^3 A + \tan\theta - \tan^2 A \tan\theta,$$

or $$\tan\theta = \tan^3 A.$$

[This property is the same as that proved in the note on the geometrical method of obtaining the cube of any number, given on p. 53 of Vol. XLIII.]

7908. (By R. KNOWLES, B.A.)—Tangents are drawn from a point $T(h, k)$ to meet the ellipse $a^2y^2+b^2x^2=a^2b^2$, centre C, in P and Q; prove that (1) $CP^2 \quad CQ^2 = \left[4a^2b^2(a^2-b^2)\,hk\,(a^2k^2+b^2h^2-a^2b^2)^{\frac{1}{2}}\right] / (a^2k^2+b^2h^2)^2$; (2) if T be on either axis, CP and CQ are equal.

Solution by Rev. T. GALLIERS, M.A.

The equation of the straight lines CP, CQ is

$$a^2b^2(b^2x^2+a^2y^2) = (b^2hx+a^2ky)^2,$$

or

$$a^4(k^2-b^2)\,y^2+2a^2b^2hkxy+b^4(h^2-a^2)\,x^2=0,$$

and if θ_1, θ_2 be the inclinations of CP, CQ to the axis of x,

$$\left.\begin{aligned}\tan\theta_1+\tan\theta_2 &= -2a^2b^2hk / a^4(k^2-b^2)\\ \tan\theta_1\tan\theta_2 &= b^4(h^2-a^2)/a^4(k^2-b^2)\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(1);$$

also, if (r, θ) be the polar coordinates, referred to C as origin, of any point on the ellipse

$$r^2 = \frac{a^2b^2}{(b^2\cos^2\theta+a^2\sin^2\theta)} = \frac{a^2b^2(1+\tan^2\theta)}{(b^2+a^2\tan^2\theta)},$$

therefore

$$CP^2 \sim CQ = \frac{a^2(a^2-b^2)(\tan^2\theta_1\sim\tan^2\theta_2)}{b^4+a^2b^2(\tan^2\theta_1+\tan^2\theta_2)+a^4\tan^2\theta_1\tan^2\theta_2}.$$

The numerator of this fraction [by substituting values $\tan\theta_1+\tan\theta_2$ and $\tan\theta_1\tan\theta_2$ from (1)] may be shown to be

$$= a^2b^2(a^2-b^2)\,2a^2b^2hk\left\{4a^4b^4(b^2h^2+a^2k^2-a^2b^2)\right\}^{\frac{1}{2}} / a^8(k^2-b^2)^2,$$

and the denominator is $= a^4b^4(b^2h^2+a^2k^2)^2 / a^8(k^2-b^2)^2$, therefore &c.

8150. (By H. GORDON DAWSON, B.A.)—Show that (1) if the roots $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, of the sextic $x^6+px^5+qx^4+rx^3+sx^2+tx+e=0$, be connected by the relation $\alpha+\beta+\gamma=\delta+\epsilon+\zeta$, then will

$$ABC+2FGH-AF^2-BG^2-CH^2=0,$$

where

$$A=2,\quad B=2\left[s-\tfrac{1}{2}pr+\tfrac{1}{4}p^2q-\tfrac{1}{16}p^4\right],\quad C=2e,$$
$$F=t,\quad G=r-\tfrac{1}{2}pq+\tfrac{1}{8}p^3,\quad H=q-\tfrac{1}{4}p^2;$$

and hence (2), if the foregoing sextic reduce to

$$x^6+q_1x^4+r_1x^3+s_1x^2+t_1x+e_1=0,$$

when its second term is removed, and if ϕ be the product of the ten distinct functions of the roots, of the type $(\alpha+\beta+\gamma-\delta-\epsilon-\zeta)^2$, then will

$$\phi = 4^9\begin{vmatrix}2 & q_1 & r_1\\ q_1 & 2s_1 & t_1\\ r_1 & t_1 & 2e_1\end{vmatrix}^2.$$

Solution by the PROPOSER.

Let the sextic be identified with the product of the two factors

$$x^3+ax^2+bx+c,\quad\text{and}\quad x^3+ax^2+b'x+c'.$$

Then $$b+b' = q-\tfrac{1}{4}p^2,\quad c+c' = r-\tfrac{1}{2}pq+\tfrac{1}{8}p^3,$$
$$bb' = s-\tfrac{1}{2}pr+\tfrac{1}{4}p^2q-\tfrac{1}{16}p^4,\quad bc'+b'c = t,\quad cc' = e,$$
Now $\begin{vmatrix} 2 & b+b' & c+c' \\ b+b' & 2bb' & bc'+b'c \\ c+c' & bc'+b'c & 2cc' \end{vmatrix} \equiv 0$, and hence the first result; as regards the second, the expressions ϕ and $\begin{vmatrix} 2 & q_1 & r_1 \\ q_1 & 2s_1 & t_1 \\ r_1 & t_1 & e_1 \end{vmatrix}^2$ both vanish when the sum of three roots = the sum of the remaining three; and, as they are both of the same degree in the roots, we may assume
$$\phi = \lambda \begin{vmatrix} 2 & q_1 & r_1 \\ q_1 & 2s_1 & t_1 \\ r_1 & t_1 & e_1 \end{vmatrix}^2.$$
If we take a *special* sextic, we easily find $\lambda = 4^9$, therefore, &c.

8031. (By E. Rutter.)—Prove that no cube number except 8, when increased by 1, can be a square.

Solution by B. Hanumanta Rau, M.A.; A. Gordon; *and others.*

If possible, let $n^3+1 = m^2$, then $n^3 = m^2-1 = (m-1)(m+1)$. But $m-1$ and $m+1$ cannot have any common factor other than two, nor can $m-1$ and $m+1$ be each a cube. Hence $(m-1)(m+1)$ cannot be a cube, unless it be the cube of 2.

7627. (By W. J. C. Sharp, M.A.)—If two tetrahedrons be such that the intersections of corresponding faces lie in the same plane, the lines joining corresponding vertices, and the planes through corresponding edges, all meet in a point, and conversely.

Solution by the Proposer.

Let $\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = 0$ be the faces of the first tetrahedron, and $l\alpha+m\beta+n\gamma+p\delta = 0$ the plane in which these meet the corresponding faces of the second; then the equations to these will be
$$l'\alpha+m\beta+n\gamma+p\delta = 0,\quad l\alpha+m'\beta+n\gamma+p\delta = 0,$$
$$l\alpha+m\beta+n'\gamma+p\delta = 0, \text{ and } l\alpha+m\beta+n\gamma+p'\delta = 0;$$
and the lines joining the corresponding vertices will be
$$(m'-m)\beta = (n'-n)\gamma = (p'-p)\delta,\quad (n'-n)\gamma = (p'-p)\delta = (l'-l)\alpha, \text{ \&c.,}$$
which meet at the point $(l'-l)\alpha = (m'-m)\beta = (n'-n)\gamma = (p'-p)\delta$, in which the planes $(l'-l)\alpha = (m'-m)\beta$, &c. (the planes through corresponding edges) meet. Again, if the point be fixed as above, it is easily shown that the planes $A\beta = 0$, &c., must be $l'\alpha+m\beta+n\gamma+p\delta = 0$, &c.; therefore all meet the corresponding planes of the tetrahedron of reference on $l\alpha+m\beta+n\gamma+p\delta = 0$.

APPENDIX.

SOME PROPERTIES OF A QUADRILATERAL IN A CIRCLE, THE RECTANGLES UNDER WHOSE OPPOSITE SIDES ARE EQUAL.

By R. Tucker, M.A.

If any circle, having a side BC of the triangle ABC as chord, cuts the sides AB, AC in C′, B′, the line C′B′ is called an *antiparallel* of BC with regard to the angle A: the locus of the mid-points of all such antiparallels (which are, of course, parallel to C′B′) is a straight line, $A\alpha$, which is called the *median antiparallel* of BC with regard to the same angle A.

It is readily proved that the three median antiparallels ($A\alpha$, $B\beta$, $C\gamma$) of a triangle cointersect in a point (K), which has been variously named the *centre of median antiparallels*, the *point de Grebe*, the *point de Lemoine*, and the *Symmedian-point* of the triangle.* In the year 1873, M. E. Lemoine gave the above definitions, and further showed that, if parallels be drawn through K to the sides of the triangle, their six intersections with the sides lie on a *circle* (A). Again, it has long been known that lines AO, BO, CO, AO′, BO′, CO′ can be drawn to points O, O′ within a triangle such that $ABO = BCO = CAO = BAO' = ACO' = CBO' = \omega$, where $\cot\omega = \cot A + \cot B + \cot C$.

In April 1881, Captain Brocard showed that a circle can be drawn through O, O′, K, the circum-centre of the triangle, and also through the three other points of intersection of the lines AO, BO, &c. (B). Subsequent writers have called this circle *Brocard's circle*, the points O, O′ *Brocard-points*, and the angle ω the *Brocard-angle* (C).† In the year 1883, I rediscovered Lemoine's circle, which, from a particular property of it, I called the "Triplicate-Ratio" Circle; and I proved that it and Brocard's circle are concentric (D). These circles, though apparently little known in England until my paper on the Triplicate-Ratio Circle

* For a further account of these points and lines I refer the reader to the Memoirs, whose titles are given in the Bibliographical Note at the end of this article. Reference is made to these throughout the present paper, thus (A), (B), &c.

† Prof. Neuberg, to whom I sent this paper in slip, informs me that the *Brocard-circle* was given first in the *Nouvelle Correspondance de Catalan*, Tom. vi., 1880.

was published, have for the last two or three years attracted considerable attention on the Continent.

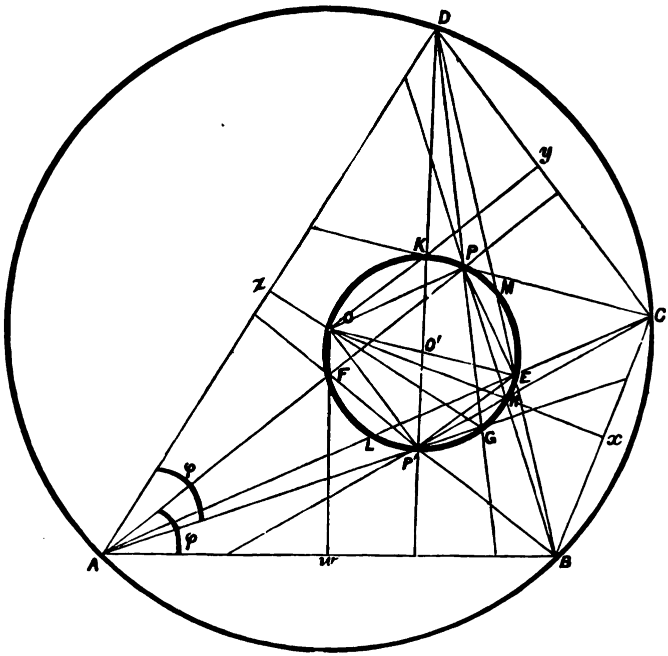


Fig. 1.

It naturally occurred to me to attempt the extension of these properties to the quadrilateral, but I was brought to a stand at the outset by the fact that the equality of angles does not involve the similarity of the figures for figures of a higher order than the triangle. Limiting the figures, however, by the restriction that they shall be circumscriptible, I have arrived at the following results, which are, I believe, new.

I find, first, the condition for the existence of a Brocard-point for the general figure. Let ABCD be a quadrilateral, with sides $AB = a$, $BC = b$, $CD = c$, $DA = d$, and suppose that a point P can be found, so that $\angle PAB = \angle PBC = \angle PCD = \angle PDA = \phi$; then, if c_1, c_2, c_3, c_4 stand for the sum of the simple cotangents of the angles A, B, C, D, and for the sum of their products taken two, three, four at a time respectively, we have $AP : BP = \sin(B-\phi) : \sin\phi$, $BP : CP = \sin(C-\phi) : \sin\phi$,

$$CP : DP = \sin(D-\phi) : \sin\phi, \quad DP : AP = \sin(A-\phi) : \sin\phi,$$

$$\therefore \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C \operatorname{cosec} D = 1 - c_2 + c_4$$
$$= \cot^4\phi - c_1\cot^3\phi + c_2\cot^2\phi - c_3\cot\phi + c_4,$$

i.e., $$\cot^4\phi - 1 - c_1\cot\phi(\cot^2\phi + 1) + c_2(\cot^2\phi + 1) = 0,$$

whence, rejecting the factor, there results

$$\cot^2\phi - c_1\cot\phi - 1 + c_2 = 0, \quad \text{or} \quad 2\cot\phi = c_1 \pm (c_1^2 - 4c_2 + 4)^{\frac{1}{2}} \text{......(1)}.$$

Now, for the future, confining the attention to the case of the quadrilateral in a circle, it will be seen that (1) takes the form

$$\cot^2\phi = 1 + \cot^2 A + \cot^2 B \quad \text{or} \quad \operatorname{cosec}^2\phi = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B \text{......(2)*}.$$

From Fig. (1) we have

$$\left.\begin{array}{c} c\sin A + d\sin(A+B) = a\sin B, \quad b\sin(A+B) + c\sin B = a\sin A \\ b\sin A - d\sin B = a\sin(A-B) \end{array}\right\} \text{...(3)}.$$

Also $2S = 2\,(\text{Area}) = (ad+bc)\sin A$

$$= \sin^2\phi\,[(a^2+b^2+c^2+d^2)\cot\phi + (b^2-d^2)\cot A - (a^2-c^2)\cot B],$$

whence, remembering (3) and (2), we get, after some reductions,†

$$\left.\begin{array}{ll} & b = a\sin B\,(\cot\phi - \cot A), \\ \text{and thence} & c = a\left[\dfrac{1-\cos A\cos B\cos(A+B)}{\sin A\sin B} - \cot\phi\sin(A+B)\right] \\ \text{and} & d = a\sin A\,(\cot\phi - \cot B) \end{array}\right\} \text{...(4)}.$$

It will be found, by multiplication and substitution, that $ac = bd$: hence this is a condition which the quadrilateral in a circle must satisfy in order that it may have Brocard-points.

By known results, if R is the circum-radius, we have

$$\left.\begin{array}{ll} & 4R^2 = (ac+bd)\,/\sin A\sin B = 2bd\,/\sin A\sin B \\ \text{therefore, by (4),} & 2R^2 = a^2(\cot\phi - \cot A)(\cot\phi - \cot B) \end{array}\right\} \text{...(5)}.$$

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* If BC, AD be produced so as to form a triangle, and if α be the Brocard-angle of this triangle, then $\operatorname{cosec}^2\alpha = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2(A+B)$, therefore $\operatorname{cosec}^2\alpha = \operatorname{cosec}^2\phi + \operatorname{cosec}^2(A+B)$. Similarly, for the other triangle, $\operatorname{cosec}^2\alpha' = \operatorname{cosec}^2\phi + \operatorname{cosec}^2(B-A)$.

† I eliminate c, d, and get $(b\operatorname{cosec} B - a\cot\phi + a\cot A)^2 = 0$, but the work is tedious. Professor Neuberg proceeds thus:

$$\frac{a}{\sin B} = \frac{PB}{\sin\phi}, \quad \frac{b}{\sin C} = \frac{PB}{\sin(C-\phi)},$$

hence $\quad PB = \dfrac{a\sin\phi}{\sin B} = \dfrac{b\sin(C-\phi)}{\sin C}, \quad$ *i.e.*, $\dfrac{a}{b\sin B} = \cot\phi - \cot C,$

and $\quad \dfrac{b}{c\sin C} = \cot\phi - \cot D, \quad \dfrac{c}{d\sin D} = \cot\phi - \cot A.$

"L'élimination de $\cot\phi$ entre 2 de ces équations donne la condition cherchée. Il serait à désirer que la question *générale* fût complètement étudiée. Vous avez deux valeurs de $\cot\phi$: laquelle faut-il prendre?" This question seems to me to be answered in the first sentence of this note.

Let the sides a, b, c, d subtend at the opposite angles of the quadrilateral the angles θ_1, θ_2, θ_3, θ_4; then, from ΔABC, we have

$$b \sin\theta_1 = a \sin(B+\theta_1) \quad \text{or} \quad b \operatorname{cosec} B = a(\cot\theta_1 + \cot B),$$

and therefore

$$\left.\begin{array}{ll} \cot\theta_1 = \cot\phi - \cot A - \cot B, \text{ similarly } & \cot\theta_2 = \cot\phi + \cot A - \cot B \\ \cot\theta_3 = \cot\phi + \cot A + \cot B, & \cot\theta_4 = \cot\phi - \cot A + \cot B \end{array}\right\} \text{.....(6)}.$$

We have then
$$\left.\begin{array}{l} \cot\theta_1 + \cot\theta_3 = 2\cot\phi = \cot\theta_2 + \cot\theta_4 \\ \Sigma\cot\theta = 4\cot\phi, \quad \Sigma\operatorname{cosec}^2\theta = 8\cot^2\phi \\ \Sigma\cot^3\theta = 4\cot\phi\,(4\cot^2\phi - 3) \end{array}\right\} \text{............(7)}.$$
and

If ω_1, ω_2, ω_3, ω_4 are the Brocard-angles of the triangles ABD, BCA, CDB, DAC, then

$$\left.\begin{array}{l} \cot\omega_1 = \cot A + \cot\theta_1 + \cot\theta_4 = 2\cot\phi - \cot A \\ \cot\omega_2 = 2\cot\phi - \cot B, \quad \cot\omega_3 = 2\cot\phi + \cot A \\ \cot\omega_4 = 2\cot\phi + \cot B \end{array}\right\} \text{.........(8)},$$

whence
$$\left.\begin{array}{l} \cot\omega_1 + \cot\omega_3 = 4\cot\phi = \cot\omega_2 + \cot\omega_4 \\ \Sigma\cot\omega = 8\cot\phi = 2\Sigma\cot\theta, \quad \Sigma\cot^2\omega = 18\cot^2\phi - 2 \end{array}\right\} \text{......(9)}.$$
and

From (2) it is seen that, since the angles are symmetrically involved, there is a second Brocard-point if there is one; let the one Brocard-point (P) be got by making $\angle BAP = \&c. = \phi$, and the other Brocard-point (P′) by making $\angle ABP' = \&c. = \phi$;* the lines thus drawn will further intersect by twos in the four points F, G, H, K: these six points, I proceed to show, lie on a circumference which I call the Brocard-circle of the quadrilateral.

For (Fig. 1) $\angle FPH = \pi - B$, and $\angle FP'H = B$, therefore F, P, P′, H are concyclic.

Again, $\angle PFP' = 2\phi$, and $\angle PGP' = \pi - 2\phi$, therefore G is on the circle, and similarly K is on the same circle; *i.e.*, the six points lie on the Brocard-circle.

From the circumcentre O, let fall perpendiculars bisecting the sides a, b, c, d in the points w, x, y, z; it is readily seen that these perpendiculars pass through F, G, H, K respectively. Now

$$\angle FP'A = \angle A = \angle wOG,$$

therefore O is on the Brocard-circle. Take E the point of intersection of the diagonals AC, BD, then the perpendicular from E on AB

$$= a/(\cot\theta_2 + \cot\theta_4) = a\tan\phi/2 = a\sigma = FW,$$

therefore OE is a diameter of the Brocard-circle..................(11).

Hence this circle bisects all chords of the circum-circle, which pass through E, and therefore it passes through the mid-points, L, M, of AC, BD.

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* P′ is the first Brocard-point and P the second Brocard-point, to employ a definition first given by Miss C. A. Scott.

From Fig. 1, it may be seen that

$$\left.\begin{aligned} AP &= d\sin\phi\,\mathrm{cosec}\,A, & AP' &= a\sin\phi\,\mathrm{cosec}\,A \\ BP &= a\sin\phi\,\mathrm{cosec}\,B, & BP' &= b\sin\phi\,\mathrm{cosec}\,B \\ CP &= b\sin\phi\,\mathrm{cosec}\,A, & CP' &= c\sin\phi\,\mathrm{cosec}\,A \\ DP &= c\sin\phi\,\mathrm{cosec}\,B, & DP' &= d\sin\phi\,\mathrm{cosec}\,B \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(12).$$

Hence may be obtained the following results, among others:

$$\left.\begin{aligned} AP\,.\,BP\,.\,CP\,.\,DP &= abcd\sin^4\phi \,/\, \sin^2 A\sin^2 B \\ &= 4R^4\sin^4\phi = AP'.\,BP'.\,CP'.\,DP' \\ AP\,.\,CP = AP'.\,CP', &\quad BP\,.\,DP = BP'.\,DP' \\ AP\,.\,BP' = 2R^2\sin^2\phi &= BP\,.\,CP' = CP\,.\,DP' = DP\,.\,AP' \\ AP'^2 + BP^2 &= a^2\sin^2\phi\,(\mathrm{cosec}^2 A + \mathrm{cosec}^2 B) = a^2 \\ BP'^2 + CP^2 = b^2, &\quad CP'^2 + DP^2 = c^2 \\ DP'^2 + AP^2 &= d^2 \end{aligned}\right\} \ldots\ldots(13).$$

If $\tau_1, \tau_2, \tau_3, \tau_4$ are the tangents from A, B, C, D to the Brocard-circle, we get

$$\left.\begin{aligned} \tau_1^2 &= AP\,.\,AF = ad\,\sigma\,\mathrm{cosec}\,A, & \tau_2^2 &= ab\,\sigma\,\mathrm{cosec}\,B \\ \tau_3^2 &= bc\,\sigma\,\mathrm{cosec}\,A, & \tau_4^2 &= cd\,\sigma\,\mathrm{cosec}\,B \end{aligned}\right\} \ldots\ldots\ldots(14).$$

Since $\qquad OD = OA, \quad \angle ODP = \angle OAP',$

and $\qquad \angle OPD + \angle OP'A = 2$ right angles,

therefore $\qquad OP = OP'$ and $EP = EP'$ ....(15).

Take ρ for the radius of the Brocard-circle, then, since

$$\angle POP' = \angle PFP' = 2\phi,$$

we have $\qquad PP' = 2\rho\sin 2\phi,$ and $AP'.\,DP = R^2 - OP^2,$

therefore $\qquad OP^2 = R^2 - ac\sin^2\phi\,\mathrm{cosec}\,A\,\mathrm{cosec}\,B,$ by (12),

$$= R^2 - 2R^2\sin^2\phi, \text{ by (4) and (5),}$$

$$\left.\begin{aligned} \therefore\; OP = R\,(\cos 2\phi)^{\frac12} = 2\rho\cos\phi, \quad \therefore\; 2\rho &= R\sec\phi\,(\cos 2\phi)^{\frac12} \\ \text{or} \qquad 4\rho^2 / R^2 &= (\cot^2 A + \cot^2 B) / (1 + \cot^2 A + \cot^2 B) \end{aligned}\right\} \ldots(16).$$

Let α denote the perpendicular from E on AB, then $\alpha = a\tan\phi / 2 = a\sigma$, see (11). Now, $\quad 2S = (ad + bc)\sin A = (ab + cd)\sin B,$

therefore $\quad 4S^2 = \sin A\sin B\,bd\,(a^2 + b^2)(a^2 + d^2) / a^2$

$$= 2R^2\sin^2 A\sin^2 B\,(a^2 + b^2)(a^2 + d^2) / a^2, \text{ by (4) and (5);}$$

but, by (2) and (4), $\quad a^2 + b^2 = 2a^2\cot\phi\,(\cot\phi - \cot A)\sin^2 B,$

and $\qquad a^2 + d^2 = 2a^2\cot\phi\,(\cot\phi - \cot B)\sin^2 A,$

therefore $\quad (a^2 + b^2)(a^2 + d^2) / a^2 = a^2 + b^2 + c^2 + d^2 = 8R^2\cot^2\phi\sin^2 A\sin^2 B.$

Hence $\qquad \dfrac{2S}{a^2 + b^2 + c^2 + d^2} = \dfrac{2^{\frac12} aR\sin A\sin B}{(a^2 + b^2)^{\frac12}(a^2 + d^2)^{\frac12}} = \dfrac{\alpha}{a};$

by a comparison with (3) of (E), it may be seen that the point E is the

analogue of the Symmedian-point of a triangle. This analogy may be still further confirmed. Through E (Fig. 2) draw lines parallel to the

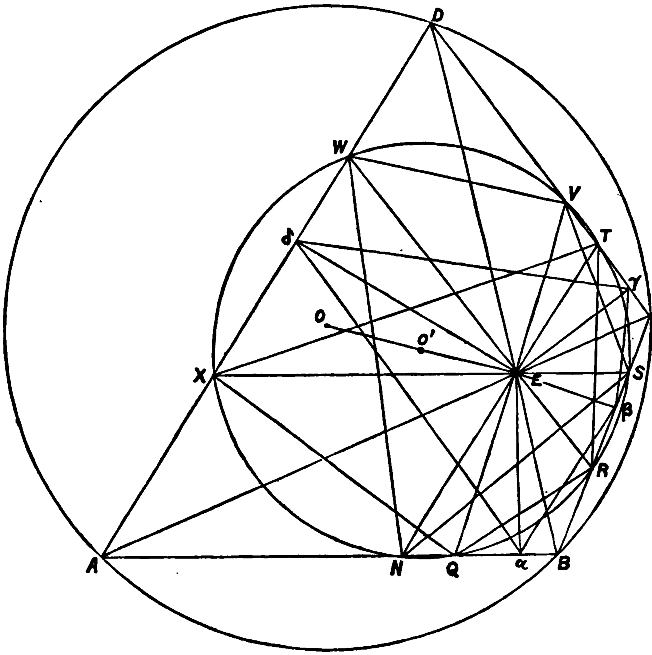

FIG. 2.

sides a, b, c, d, meeting the sides taken in this order, in the points N, Q; R, S; T, V; W, X. These lines clearly pass through some one of the points F, G, H, Q. If now O′ be taken as the centre of the Brocard-circle, then the projection of O′X on AB

$$= \left(\frac{2\cot\phi + \cot A - \cot B}{4\cot\phi} - \frac{\cot A}{2\cot\phi}\right) a$$

$$= (2\cot\phi - \cot A - \cot B)\, a\, /\, (4\cot\phi).$$

Similarly this will be found to be the projection of O′S on the same line, therefore O′X = O′S; and O′ is the centre of a circle passing through the eight points obtained above: a circle in some respects corresponding to the triplicate-ratio circle of a triangle. The following relations may be noted:

$$\left.\begin{aligned} EX\sin A &= d\sigma = EW\sin B, & EV\sin A &= c\sigma = ET\sin B \\ ER\sin A &= b\sigma = ES\sin B, & EN\sin A &= a\sigma = EQ\sin B \end{aligned}\right\} \ldots(17).$$

$$\left.\begin{array}{l}\text{Again,} \qquad NQ = a\sigma(\cot A + \cot B) \\ \text{and similar expressions hold for RS, TV, WX}\end{array}\right\} \text{......................(18)},$$

$$\left.\begin{array}{l}\text{so that} \qquad NQ : RS : TV : WX \\ = a\sin(A+B) : b\sin(B-A) : c\sin(A+B) : d\sin(B-A)\end{array}\right\} \text{...(19)}.$$

Again, if for a moment $\angle ANX = \theta$,

$$\frac{\sin(A+\theta)}{\sin\theta} = \frac{AN}{AX} = \frac{EX}{EN} = \frac{d}{a} = \frac{AD}{AB} = \frac{\sin(A+\theta_1)}{\sin\theta_1},$$

therefore $\theta = \theta_1$, *i.e.*, $\angle ANX\,(= \angle AWQ) = \angle ADB$; hence B, N, X, D are concyclic, and QW parallel to BD.

Similarly B, R, V, D are concyclic and ST parallel to BD.

Again, $\angle BSQ = \theta_2$, hence A, Q, S, C are concyclic, and NR parallel to AC; also $\angle WTD = \theta_3$, hence A, C, T, W are concyclic, and XV parallel to AC.

$$\left.\begin{array}{l}\text{Further} \qquad XN = WT = \sigma\,.\,BD\cot A = RV = SQ \\ \text{and} \qquad XQ = SN, \quad XT = WN, \quad WV = RT, \ \&c.\end{array}\right\} \text{............(20)}.$$

Since $\quad 2\Delta ANX = ad\sigma^2\cot^2 A\sin A, \quad 2\Delta BQS = ab\sigma^2\cot^2 B\sin B,$

$2\Delta CRV = bc\sigma^2\cot^2 A\sin A, \quad 2\Delta DTW = cd\sigma^2\cot^2 B\sin B,$

the sum of these four triangles equals

$$(ad+bc)\,\sigma^2\cot^2 A\sin A + (ab+cd)\,\sigma^2\cot^2 B\sin B$$

$$= 2R^2\sigma(\sin^2 A\cos^2 B + \cos^2 A\sin^2 B) = 2R^2\sin^2 A\sin^2 B\cot 2\phi,$$

therefore area of octogon* (XNQSRVTWX)

$$= 2R^2\sin^2 A\sin^2 B\,\mathrm{cosec}\,2\phi \text{.....................(21)}.$$

Since the projection of O′X on SX is

$$(2\cot\phi - \cot A - \cot B)\,a\,/\,(4\cot\phi) = a\,(\cot\phi + \cot\theta_1)\,/\,(4\cot\phi),$$

and perpendicular from O′ on same line $= (\cot\theta_1 \sim \tan\phi)\,/\,4$, therefore, if ρ' = radius of this circle,

$$16\rho'^2 = a^2\tan^2\phi\,[\cot^2\phi + \cot^2\theta_1 + \cot^2\phi\cot^2\theta_1 + 1]$$

$$= a^2\tan^2\phi\,(1+\cot^2\phi)(1+\cot^2\rho_1) = a^2\sec^2\phi\,\mathrm{cosec}^2\theta_1 = 4R^2\sec^2\phi,$$

therefore $\qquad \rho' = R\sec\phi\,/\,2 = R\,.\,AF\,/\,AB$........................(22).

From (16) and (22), $\qquad \rho^2 + \rho'^2 = R^2/2$ ....(23).

If we suppose $\angle BNS = \psi$, then, because $\angle BQS = \angle ANX = \theta_1$,

$$NQ = a\sigma(\cot\psi - \cot\theta_1),$$

but, by (18), $\qquad NQ = a\sigma(\cot A + \cot B),$

hence, by (6), we have $\psi = \phi$.

It follows, therefore, that WN, NS, SV, VW, and also that XT, TR, RQ, QX, make the same angle, ϕ, with their respective sides; and it is readily proved that $\angle W = \angle D$, and $WV : WN = c : d$, therefore the

* The figure, it will be observed, is not an octagon in the ordinary sense; the word "octogon" was suggested to me by Captain Brocard.

two inscribed quadrilaterals SVWN, RTXQ are similar to ABCD and therefore to one another. The ratio of similitude

$= \text{WV} : \text{CD} = \text{ET} \sin \text{B} \operatorname{cosec} \phi : \text{CD} = \sigma \operatorname{cosec} \phi : 1 = \sec \phi : 2 \ldots (24),$

and the two inscribed quadrilaterals are equal.

Since $\angle \text{QXE} = \angle \text{XQA} = \phi$, therefore E corresponds to a first Brocard-point of QTXR, and to a second Brocard-point of SVWN.

Now, put $\angle \text{AP}'\text{X} = \psi$, then

$$\frac{\sin \psi}{\sin (\psi + \phi)} = \frac{\text{AX}}{\text{AP}'} = \frac{\text{EN}}{\text{AP}'} = \frac{1}{2 \cos \phi},$$

therefore $\psi = \phi$, and therefore $\angle \text{DXP}' = 2\phi$, and $\angle \text{TXP}' = \phi$; hence P′ is the second Brocard-point of XTRQ, and similarly P is the first Brocard-point of SVWN.

Hence it follows that O′EP, O′EP′ are Brocard-circles of SVWN and PTRQ respectively; they cut one another at an angle $= \pi - 2\phi$, *i.e.*, X′E, PE are tangents to them respectively at E; also, if ϵ, ϵ' are the points corresponding to E, then $\text{P}\epsilon$, $\text{E}\epsilon$; $\text{P}'\epsilon'$, $\text{E}\epsilon'$ are tangents to the aforesaid Brocard-circles.

Since $\angle \text{AQX} = \phi = \angle \text{AP}'\text{X}$, the circles described round the triangles AQX, BQR, CRT, DTX pass through P′, and similarly circles round ANW, BNS, CSV, DVW pass through P. Their radii, taken in the above order, are $ap \operatorname{cosec} \text{A}$, $bp \operatorname{cosec} \text{B}$, $cp \operatorname{cosec} \text{A}$, $dp \operatorname{cosec} \text{B}$; $dp \operatorname{cosec} \text{A}$, $ap \operatorname{cosec} \text{B}$, $bp \operatorname{cosec} \text{A}$, $cp \operatorname{cosec} \text{B}$; where $4p \cos \phi = 1$.

If the points P, P′ are projected on the four sides, the product of their projections reckoned from A along $\text{AB} = \text{AP} \cos \phi \,.\, \text{AP}' \cos (\text{A} - \phi) = \text{AP} \cos (\text{A} - \phi) \,.\, \text{AP}' \cos \phi =$ product of projections reckoned from A along AD; hence a circle will pass through the points of projection on AB, AD, and have its centre at the mid-point of PP′: this is evidently true for the similar points on the other sides; therefore this circle passes through the eight feet of projections from P, P′ on the sides.

If ρ'' is the radius of this circle,

$$4\rho''^2 \sin^2 \text{A} \operatorname{cosec}^2 \phi = [d \cos \phi - a \cos (\text{A} - \phi)]^2 + [d \sin \phi + a \sin (\text{A} - \phi)]^2$$
$$= a^2 + d^2 - 2ad \cos \text{A} = \text{BD}^2$$

$$\therefore \quad 2\rho'' = \text{BD} \sin \phi \operatorname{cosec} \text{A} = \text{AC} \sin \phi \operatorname{cosec} \text{B}$$
$$= (\text{AC} \,.\, \text{BD} \operatorname{cosec} \text{A} \operatorname{cosec} \text{B})^{\frac{1}{2}} \,.\, \sin \phi = (2bd \operatorname{cosec} \text{A} \operatorname{cosec} \text{B})^{\frac{1}{2}} \,.\, \sin \phi$$
$$= 2\text{R} \sin \phi, \text{ by } (5),$$

i.e., $$\rho'' = \text{R} \sin \phi \quad \ldots\ldots (25).$$

If p_1, p_1'; &c., are the perpendiculars from P, P′, on AB, &c., then $p_1 p_1' = \text{AP} \,.\, \text{BP}' \sin^2 \phi = 2\text{R}^2 \sin^4 \phi$, by (13); hence P, P′ are the foci of an ellipse inscribed in ABCD, major axis $(= 2\text{R} \sin \phi)$ in PP′ produced, minor axis $= 2\sqrt{2}\,\text{R} \sin^2 \phi$, and eccentricity $= (\cos 2\phi)^{\frac{1}{2}}$.

Let the feet of the perpendiculars from E on the sides a, b, c, d be denoted by α, β, γ, δ; then

$$\alpha\beta = \text{EB} \sin \text{B}, \quad \beta\gamma = \text{EC} \sin \text{A}, \quad \gamma\delta = \text{ED} \sin \text{B}, \quad \delta\alpha = \text{EA} \sin \text{A}.$$

Then $$\alpha\delta + \beta\gamma = \text{AC} \sin \text{A} = k \text{ (say)} = \text{BD} \sin \text{B} = \alpha\beta + \gamma\delta;$$

hence a circle can be inscribed in $\alpha\beta\gamma\delta$. If r_1 is the radius of this circle,

Area of $\alpha\beta\gamma\delta = kr_1$;

but 2 Area $=$ Eα . Eβ sin B + Eβ . Eγ sin A + Eγ . Eδ sin B + Eδ . Eα sin A,

and $$E\alpha = a / (\cot\theta_2 + \cot\theta_4) = a\sigma.$$

Similarly $$E\beta = b\sigma, \quad E\gamma = c\sigma, \quad E\delta = d\sigma,$$

therefore $$2\text{ Area} = \sigma^2[(ad+bc)\sin A + (ab+cd)\sin B] = 4S\sigma^2$$
$$= 4R^2\sigma\sin^2 A\sin^2 B,$$

and $k^2 = AC\sin A \,.\, BD\sin B = 2bd\sin A\sin B = 4R^2\sin^2 A\sin^2 B$,

therefore $$r_1 = 2R^2\sigma\sin^2 A\sin^2 B / (2R\sin A\sin B)$$
$$= \sigma \,.\, R\sin A\sin B \quad\text{.......................................(26).}$$

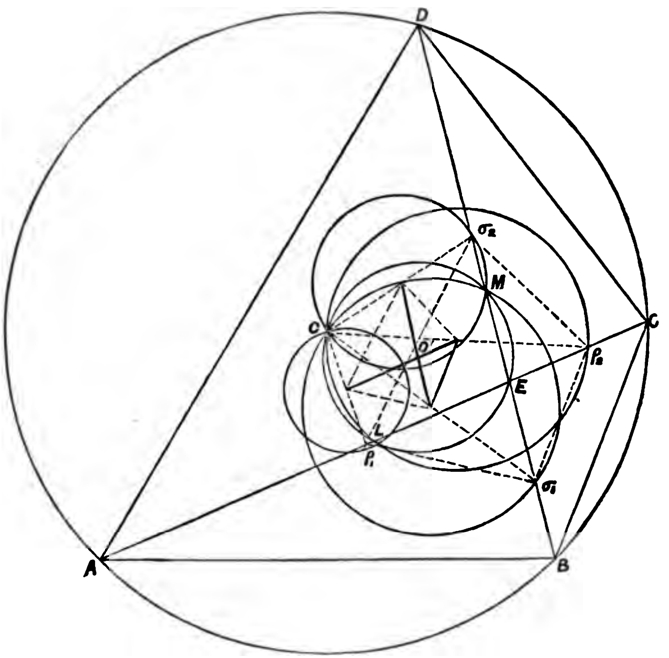

Fig. 3.

The perpendicular from E on $\alpha\beta = E\alpha\sin\theta_3 = \sigma bd / (2R)$
$$= \sigma R\sin A\sin B = r_1,$$

therefore E is the centre of the circle; as is also evident from the fact that Eα, Eβ, Eγ, Eδ bisect the angles which are respectively equal to $2\theta_3$, $2\theta_1$, $2\theta_2$, $2\theta_4$. Also, since Eα cos αBE $= k\alpha$ cos αBE, where $k = 2S / (a^2+b^2+c^2+d^2)$,

and similar values for $E\beta$, $E\gamma$, $E\delta$, and it is evident that

$$a \cos aBE + d \cos EDA = b \cos EBC + c \cos EDC,^*$$

therefore E is the mean-centre of $\alpha\beta\gamma\delta$ . . . (27) (the exactly analogous property for a triangle is taken by Dr. CASEY for the determination of the Symmedian-point, K, of a triangle).

Returning to the quadrilateral ABCD (Fig. 3), since

$$BE : ED = \sin\theta_1 \sin\theta_2 : \sin\theta_3 \sin\theta_4 = a^2 : d^2,$$

therefore the Symmedian-point (ρ_1) of $\triangle$ABD is on AC, similarly the Symmedian-point (ρ_2) of BCD is on the same line; also the Symmedian-points (σ_1, σ_2) of ABC, ADC lie on BD [*cf.* (E) (iv.)]. The lines $O\rho_1$, $O\rho_2$, $O\sigma_1$, $O\sigma_2$ are the diameters of the Brocard-circles of the triangles ABD, BCD, ABC, ACD respectively. The centres of the four Brocard-circles are readily seen to lie two and two on straight lines, parallel to AC, BD, which intersect in O′, and the circles themselves intersect two and two on the diagonals, AC, BD, at their mid-points, *i.e.*, where the Brocard-circle of the quadrilateral meets the diagonals. But I defer the further consideration of these circles for the present. *February*, 1885.

NOTE. — (A) M. E. LEMOINE, "Sur quelques propriétés d'un point remarquable d'un triangle" (Association Française pour l'Avancement des Sciences—Congrès de Lyon, Août, 1873).

(B) M. H. BROCARD, "Etude d'un nouveau cercle du plan du triangle" (Association Française, &c.; Congrès d'Alger, Avril, 1881).

(C) Professor J. NEUBERG, "Sur le centre des médianes antiparallèles" (*Mathesis*, Oct., Nov., Dec., 1881).

(D) M. MAURICE D'OCAGNE, "Sur les Symédianes" (*Nouvelles Annales*, Oct. 1883, Jan. 1884). This writer obtained numerous results already got by M. Lemoine, without being aware of that fact.

(E) R. TUCKER, The "Triplicate-Ratio" Circle (*Quarterly Journal of Mathematics*, Vol. xix., No. 76, date of paper May, 1883).

Dr. CASEY, "A Sequel to the first Six Books of the Elements of Euclid." (3rd edition, 1884, pp. 163—172. Dr. Casey's attention was first drawn to the circles by the present writer; he has proved their properties in an elegant form, and added results of his own.)

A list of continental writers upon the subject and of papers is given in the "Appendix" to the *Proceedings of the London Mathematical Society*, vol. xiv., pp. 316—321; vol. xv., pp. 278—281.

I have not succeeded in getting a geometrical construction by means of which the quadrilateral treated of in this article can be readily obtained, but, by making in (ii.) cot A $= \frac{3}{5}$, cot B $= -\frac{2}{5}$, we get cot $\phi = \frac{16}{13}$, it will be found, on plotting these angles, that they come out very approximately equal 58°, 112°, and 38° respectively; also, in this case, by (5), AB comes out equal to a side of the inscribed square of the circle.

[A very condensed abstract of the above results was published in the *Educational Times* for March, 1885. After reading this, Mr. W. S. M'CAY, Fellow of Trinity College, Dublin, sent me the following remarks, which present the subject in a different light:—

Let the vertices of a quadrilateral (Q) inscribed in circle form a harmonic row, and let q be a homothetic quadrilateral having its four vertices

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* I adopt this proof of Professor NEUBERG's as preferable to my own.

on the diagonals of Q. The sides of q intersect those of Q in 16 points. Four of these are at infinity. Four are on the radical axis of the circum-circles of Q and q. The other eight are on one circle, and the two sets of four alternates are vertices of two *equal* quadrilaterals Q_1, Q_2, similar to Q, and with sides equally inclined to the corresponding sides of Q. The common circum-centre of the two quadrilaterals Q_1, Q_2 bisects the distance between the circum-centres of Q and q.

The lines joining corresponding vertices of Q_1, Q_2 give a quadrilateral q' homothetic to Q', the quadrilateral formed by tangents at vertices of Q to its circum-circle, the centre of homotheticism being the same as for Qq; namely K, the intersection of diagonals of Q.

When q' shrinks into the point K, we have the case of parallels through K to the circum-tangents of Q cutting the sides of Q in eight points on a circle whose radius is $R \tan\omega$, R being circum-radius of Q and $\cot\omega = \dfrac{a^2+b^2+c^2+d^2}{4 \text{ area } Q}$, a, b, c, d being sides of Q (1). Also, in this case, the sides of Q_1, Q_2 are perpendicular to those of Q, and K is the common centre of *four* rectangles, each rectangle standing on a side of Q and with its other two vertices on the adjacent sides of Q.

When q shrinks into the point K, we have parallels to the sides of Q through K cutting sides in eight points on a circle whose centre is mid-point of HK, H being circum-centre of Q.

As q varies, the intersection of diagonals K_1 of Q_1 (and K_2 of Q_2) describes the line HK.

I believe all these results would hold for a polygon in a circle, if the sides were so related that there existed a point whose distances from the sides were proportional to the sides. This would be true if the polygon were in perspective with the polygon formed by tangents at vertices.

I have just noticed that circle mentioned above (1) makes intercepts on the sides proportional to the cosines of the angles subtended at any point of the circum-circle of Q by the sides a, b, c, d.]

Professor Neuberg sends several interesting results, but, as these will be given in *Mathesis*, I extract only the following brief remarks:—

1. Construction for point P, which agrees with my own, founded upon the fact that $\angle APB = \pi - ABC$, &c. But I do not see that the construction is of practical use.

2. The quadrilateral ABCD might be called a ***harmonic quadrilateral.***

3. The perpendiculars let fall from P on a, b, c, d are proportional to

$$d/\sin A, \quad a/\sin B, \quad b/\sin C, \quad c/\sin D.$$

"Ce qui est une nouvelle analogie avec les points de Brocard du triangle, dont les distances aux côtés du triangle sont proportionnelles à

$$b/c, \quad c/a, \quad a/b, \quad \text{ou} \quad b/\sin C, \quad c/\sin A, \quad a/\sin B, \ \text{\&c.''}$$

4. $$E\alpha = EB \sin EBA = EB \sin DCA, \quad E\beta = EB \sin DAC,$$

therefore $\dfrac{E\alpha}{E\beta} = \dfrac{\sin DCA}{\sin DAC} = \dfrac{d}{c} = \dfrac{a}{b}$, therefore $\dfrac{\alpha}{a} = \dfrac{\beta}{b}$;

also $$a\alpha + b\beta + c\gamma + d\delta = 2S,$$

therefore $\dfrac{\alpha}{a} = \dfrac{\beta}{b} = \ldots = \ldots = \dfrac{2S}{a^2+b^2+c^2+d^2}$, *cf.* (27).

APPENDIX II.

AN APPLICATION OF DETERMINANTS TO THE SOLUTION OF CERTAIN TYPES OF SIMULTANEOUS EQUATIONS.

By Rev. T. C. Simmons, M.A.

In a brief note on p. 43 of Vol. xlii., of the *Educational Times Reprint*, I have indicated a method by which, with the aid of determinants, certain equations in x, y, z can be immediately solved. It is proposed in this paper to gather up what was there said, and to give some further applications.

I. Take first the equations

$$yz(y+z-x) = a, \quad zx(z+x-y) = b, \quad xy(x+y-z) = c \text{ ......(i.)}.$$

We may, if we like, adopt the method of solution given on p. 26 of Vol. xl. of the *Reprint*, which is as follows. Put

$$x+y+z = p, \quad yz+zx+xy = q, \quad xyz = r;$$

then, after some manipulation, we can obtain a cubic equation in r, leading to the values of p and q; x, y, z are then the roots of the cubic equation $x^3-px^2+qx-r = 0$. It will be seen that this method not only involves the solution of *two* cubics, but gives no means of distinguishing finally between the individual values of x, y, z.

A very curious method of solving the equations for the particular values $a = \frac{15}{2}$, $b = 3$, $c = \frac{3}{4}$, is given on p. 172 of Wharton's *Key to Examples in Algebra*. It consists in putting $y = mx$, $z = nx$, and thence obtaining an *indeterminate* equation in m and n, leading to *trial* values of x, y, z.

Both of the above methods are unsatisfactory. A better plan is to obtain from equations (1) and (2), the new equation

$$\frac{a}{yz} + \frac{b}{zx} = 2z,$$

which, on putting $x = \dfrac{bc}{x'}$, $y = \dfrac{ca}{y'}$, $z = \dfrac{ab}{z'}$, becomes $z'^2(x'+y') = 2a^2b^2c$.

This, combined with the two similarly obtained equations

$$x'^2(y'+z') = 2b^2c^2a, \quad y'^2(z'+x') = 2c^2a^2b,$$

can easily be solved by the ordinary processes, or as in equation (vi.) below.

The preceding remarks are given merely by way of introduction, and in order to show how clumsy and uncertain, for an equation of this kind, are the ordinary methods compared with the following one.

II. Write the equations thus—

$$-\left(\frac{a}{xyz}+1\right)x+y+z=0 \quad \text{..............................(1)},$$

$$x-\left(\frac{b}{xyz}+1\right)y+z=0 \quad \text{..................................(2)},$$

$$x+y+\left(\frac{c}{xyz}+1\right)z=0 \quad \text{..................................(3)}.$$

Then, *eliminating* x, y, z *singly, while retaining the product* xyz, we obtain

$$\begin{vmatrix} -\left(\frac{a}{xyz}+1\right), & 1, & 1 \\ 1, & -\left(\frac{b}{xyz}+1\right), & 1 \\ 1, & 1, & -\left(\frac{c}{xyz}+1\right) \end{vmatrix} = 0,$$

or $$4x^3y^3z^3-(ab+bc+ca)\,xyz-abc=0.$$

If r be any root of this equation, then from (1), (2), (3), we can find the ratios of x, y, z in the form $x=mz$, $y=nz$; whence $mnz^3=r$, determining z, and consequently x and y.

If the equations had been proposed in the more general form

$$yz\,(lx+my+nz)=a, \quad zx\,(l'x+m'y+n'z)=b, \quad xy\,(l''x+m''y+n''z)=c,$$

it will be seen that the method is equally applicable, whereas the ordinary methods apparently cannot be applied at all.

III. The last remark applies to equations of the type

$$\left.\begin{aligned} ax^n+by^n+cz^n &= kx^{n+m}\,y^m z^m \\ a'x^n+b'y^n+c'z^n &= k'y^{n+m}\,z^m x^m \\ a''x^n+b''y^n+c''z^n &= k''z^{n+m}x^m y^m \end{aligned}\right\}.$$

As before, we eliminate x^n, y^n, z^n individually, thereby finding the value of xyz, whence x, y, z follow.

As particular examples may be given

$$\left.\begin{aligned} ax+by+cz &= kx^2yz \\ a'x+b'y+c'z &= k'y^2zx \\ a''x+b''y+c''z &= k''z^2xy \end{aligned}\right\}\text{...(ii.)}, \qquad \left.\begin{aligned} \frac{a}{x}+\frac{b}{y}+\frac{c}{z} &= kyz \\ \frac{a'}{x}+\frac{b'}{y}+\frac{c'}{z} &= k'zx \\ \frac{a''}{x}+\frac{b''}{y}+\frac{c''}{z} &= k''xy \end{aligned}\right\}\text{... (iii.)}.$$

Similar examples are

$$\left.\begin{aligned} ax+by+cz &= kxyz \\ a'x+b'y+c'z &= k'y^2z \\ a''x+b''y+c''z &= k''yz^2 \end{aligned}\right\}\text{...(iv.)}, \qquad \left.\begin{aligned} \frac{a}{x}+\frac{b}{y}+\frac{c}{z} &= ky^2z \\ \frac{a'}{x}+\frac{b'}{y}+\frac{c'}{z} &= k'xyz \\ \frac{a''}{x}+\frac{b''}{y}+\frac{c''}{z} &= k''xy^2 \end{aligned}\right\}\text{......(v.)}.$$

In the last two, we find by the determinant, not xyz, but yz and xy^2z respectively.

IV. In the preceding put

$$a = b' = c'' = 0, \quad n = -1, \quad m = 1 - r,$$

and we have
$$\left.\begin{aligned} x^r(cy + bz) &= ky^{2-r}z^{2-r} \\ y^r(a'z + c'x) &= k'z^{2-r}x^{2-r} \\ z^r(b''x + a''y) &= k''x^{2-r}y^{2-r} \end{aligned}\right\},$$

or the equivalent form
$$\left.\begin{aligned} cy + bz &= kx^m y^{2+m}z^{2+m} \\ a'z + c'x &= k'y^m z^{2+m}x^{2+m} \\ b''x + c''y &= k''z^m x^{2+m}y^{2+m} \end{aligned}\right\},$$

particular examples of which are

$$\left.\begin{aligned} x^2(cy + bz) &= k \\ y^2(a'z + c'x) &= k' \\ z^2(b''x + a''y) &= k'' \end{aligned}\right\} \ldots\text{(vi.)}, \qquad \left.\begin{aligned} cy + bz &= ky^2z^2 \\ a'z + c'x &= k'z^2x^2 \\ b''x + a''y &= k''x^2y^2 \end{aligned}\right\} \ldots\ldots\text{(vii.)}.$$

Instances of a similar type are

$$\left.\begin{aligned} yz\,(ay^2 + bz^2) &= px \\ zx\,(cz^2 + dx^2) &= qy \\ xy\,(ex^2 + fy^2) &= rz \end{aligned}\right\} \ldots\text{(viii.)}, \qquad \left.\begin{aligned} x^3(ay^2 + bz^2) &= pyz \\ y^3(cz^2 + dx^2) &= qzx \\ z^3(ex^2 + fy^2) &= rxy \end{aligned}\right\} \ldots\ldots\text{(ix.)}.$$

Of the last two, the former is solved by dividing respectively by yz, zx, xy; then finding xyz by the elimination of x^2, y^2, z^2. The latter by multiplying respectively by yz, zx, xy; then finding xyz by the elimination of x^2y^2, y^2z^2, z^2x^2. Any number of similar examples might easily be instanced.

V. Still more general is the type

$$\left.\begin{aligned} ax^m y^n + by^m z^n + cz^m x^n &= kx^{m+l}y^{n+l}z^l \\ a'x^m y^n + b'y^m z^n + c'z^m x^n &= k'x^l y^{m+l}z^{n+l} \\ a''x^m y^n + b''y^m z^n + c''z^m x^n &= k''x^{n+l}y^l z^{m+l} \end{aligned}\right\}.$$

Writing in the form $x^m y^n[a - k(xyz)^l] + by^m z^n + cz^m x^n = 0$, &c.,

we determine, first xyz; then the ratios $x^m y^n : y^m z^n : z^m x^n$; then by raising to suitable powers the ratios $x : y : z$; lastly, x, y, z absolutely.

Putting $m = 2$, $n = 1$, $l = t + 1$, $a = b' = c'' = 0$, we get the type

$$\left.\begin{aligned} x^2[p + (xyz)^t z^3] &= \lambda yz \\ y^2[q + (xyz)^t x^3] &= \mu zx \\ z^2[r + (xyz)^t y^3] &= \nu xy \end{aligned}\right\}.$$

The solution is effected by multiplying by y, z, x respectively. The determinant gives the value of $(xyz)^{t+1}$, and thence the ratios of x^2y, y^2z, z^2x. The values of x, y, z easily follow.

Again, putting $m = n = 1$, $l = t + 1$, $a = b' = c'' = 0$,

the type
$$\left.\begin{aligned} ax + by &= x^2y^2(xyz)^t \\ cy + dz &= y^2z^2(xyz)^t \\ ez + fx &= z^2x^2(xyz)^t \end{aligned}\right\}.$$

Similarly for other particular values of m, n, &c. It is also clear that instead of $(xyz)^t$ we may substitute in the above $x^\alpha y^\beta z^\gamma$.

As particular examples may be given

$$\left.\begin{aligned} x^2(p+z^3) &= ayz \\ y^2(q+x^3) &= bzx \\ z^2(r+y^3) &= cxy \end{aligned}\right\}\ldots\text{(x.)},\qquad \left.\begin{aligned} x^2\left(p+\frac{1}{x^3y^3}\right) &= ayz \\ y^2\left(q+\frac{1}{y^3z^3}\right) &= bzx \\ z^2\left(r+\frac{1}{z^3x^3}\right) &= cxy \end{aligned}\right\}\ldots\ldots\ldots\text{(xi.)}.$$

Instances of a similar kind are

$$\left.\begin{aligned} x^2\,(p+1) &= ayz \\ y^2\left(q+\frac{x^3}{z^3}\right) &= bzx \\ z^2\left(r+\frac{y^3}{z^3}\right) &= cxy \end{aligned}\right\}\ldots\text{(xii.)},\qquad \left.\begin{aligned} x^2\,(p+x^3) &= ayz \\ y^2\left(q+\frac{x^6}{z^3}\right) &= bzx \\ z^2\left(r+\frac{x^3y^3}{z^3}\right) &= cxy \end{aligned}\right\}\ldots\ldots\text{(xiii.)},$$

which may be increased to any extent.

VI. The method obviously applies also to

$$\begin{aligned} a\,x^l y^m z^n + b\,x^p y^q z^r + c\,x^g y^h z^k &= x^{l+\alpha} y^{m+\beta} z^{n+\gamma}, \\ a'x^l y^m z^n + b'x^p y^q z^r + c'x^g y^h z^k &= x^{p+\alpha} y^{q+\beta} z^{r+\gamma}, \\ a''x^l y^m z^n + b''x^p y^q z^r + c''x^g y^h z^k &= x^{g+\alpha} y^{h+\beta} z^{k+\gamma}, \end{aligned}$$

for by the determinant we find $x^\alpha y^\beta z^\gamma$; thence the ratios

$$x^l y^m z^n : x^p y^q z^r : x^g y^h z^k;$$

whence x, y, z.

The above is equivalent to the form

$$\begin{aligned} a\,x^\alpha y^\beta z^\gamma + b\,x^{p-l} y^{q-m} z^{r-n} + c\,x^{g-l} y^{h-m} z^{k-n} &= d, \\ a'x^\alpha y^\beta z^\gamma + b'x^{g-p} y^{h-q} z^{k-r} + c'x^{l-p} y^{m-q} z^{n-r} &= d', \\ a''x^\alpha y^\beta z^\gamma + b''x^{l-g} y^{m-h} z^{n-k} + c''x^{p-g} y^{q-h} z^{r-k} &= d''. \end{aligned}$$

Or, more symmetrically,

$$\begin{aligned} a\,x^\alpha y^\beta z^\gamma + b\,x^{\lambda} y^{\mu} z^{\nu} + c\,x^{-\lambda''} y^{-\mu''} z^{-\nu''} &= d, \\ a'x^\alpha y^\beta z^\gamma + b'x^{\lambda'} y^{\mu'} z^{\nu'} + c'x^{-\lambda} y^{-\mu} z^{-\nu} &= d', \\ a''x^\alpha y^\beta z^\gamma + b''x^{\lambda''} y^{\mu''} z^{\nu''} + c''x^{-\lambda'} y^{-\mu'} z^{-\nu'} &= d'', \end{aligned}$$

subject to the conditions

$$\lambda+\lambda'+\lambda'' = \mu+\mu'+\mu'' = \nu+\nu'+\nu'' = 0.$$

An instance (chosen at haphazard) of such a set of equations is

$$\left.\begin{aligned} a\,x^2y^5z^7 + b\,x^{\frac{1}{3}} y^{-2} z^{\frac{3}{4}} + c\,x^2 y^{-5} z^{-\frac{1}{2}} &= d \\ a'x^2y^5z^7 + b'x^{\frac{5}{3}} y^{-3} z^{-\frac{5}{4}} + c'x^{-\frac{1}{3}} y^2 z^{-\frac{3}{4}} &= d' \\ a''x^2y^5z^7 + b''x^{-2} y^5 z^{\frac{1}{2}} + c''x^{-\frac{5}{3}} y^3 z^{\frac{5}{4}} &= d'' \end{aligned}\right\}\ldots\ldots\ldots\ldots\text{(xiv.)}.$$

VII. The following are miscellaneous examples:—

(xv.)
$$\left.\begin{aligned} ax + by + cz &= kx^{n+1} \\ a'x + b'y + c'z &= k'x^n y \\ a''x + b''y + c''z &= kx^n z \end{aligned}\right\}.$$

The determinant is

$$\begin{vmatrix} a - kx^n, & b, & c \\ a', & b' - k'x^n, & c' \\ a'', & b'', & c'' - k''x^n \end{vmatrix} = 0,$$

whence x, y, z.

(xvi.)
$$\left.\begin{aligned} x\ (ay + bz) &= z \\ y^2 (cz + dx) &= zx \\ z\ (ex + fy) &= x \end{aligned}\right\}.$$

Writing in the form $-\frac{yz}{y} + bzx + axy = 0$, &c., we eliminate xy, yz, zx, thereby determining y^{-1}; thence the ratios of x, y, z, &c.

(xvii.)
$$\left.\begin{aligned} y\ (ax + by + cz\ ) &= 1 \\ z\ (a'x + b'y + c'z\ ) &= 1 \\ yz\ (a''x + b''y + c''z) &= x \end{aligned}\right\}.$$

Write
$$\left.\begin{aligned} ax + by + \left(c - \frac{1}{yz}\right) z &= 0 \\ a'x + \left(b' - \frac{1}{yz}\right) y + c'z &= 0 \\ \left(a'' - \frac{1}{yz}\right) x + b''y + c''z &= 0 \end{aligned}\right\}.$$

By the determinant yz is obtained; thence x, y, z separately.

(xviii.)
$$\left.\begin{aligned} x^2 + by + cz &= \frac{x}{y} \\ a'x + xy + c'z &= 1 \\ a''x + b''y + zx &= \frac{z}{y} \end{aligned}\right\}.$$

Write
$$\left.\begin{aligned} x\left(x - \frac{1}{y}\right) + by + cz &= 0 \\ a'x + \left(x - \frac{1}{y}\right) y + c'z &= 0 \\ a''x + b''y + \left(x - \frac{1}{y}\right) z &= 0 \end{aligned}\right\}.$$

Whence the value of $x - \frac{1}{y}$; then the ratios $x : y : z$; then a *quadratic* for x.

(xix.)
$$\left.\begin{aligned} ax+bz &= \frac{y}{x}(cy+dz) = 1+\frac{y}{z} \\ ex+fy &= 1+\frac{z}{y} \end{aligned}\right\}.$$

Write
$$\left.\begin{array}{lllll} -\left(\frac{1}{y}+\frac{1}{z}\right)y+ & ax & + & bz & = 0 \\ cy & -\left(\frac{1}{y}+\frac{1}{z}\right)x+ & & dz & = 0 \\ fy & + & ex & -\left(\frac{1}{y}+\frac{1}{z}\right)z & = 0 \end{array}\right\}.$$

The determinant enables us to find $\frac{1}{y}+\frac{1}{z}$; thence follow the ratios of x, y, z, &c.

(xx.)
$$z = ax^2+by = cx^3+dxy = x(ex+fy+gz).$$

Write
$$\left.\begin{array}{l} cx\,.\,x^2+d\,.\,xy - z = 0 \\ a\,.\,x^2+\frac{b}{x}\,.\,xy - z = 0 \\ e\,.\,x^2+f\,.\,xy+(gx-1)\,z = 0 \end{array}\right\},$$

then
$$\begin{vmatrix} cx, & d, & 1 \\ a, & \frac{b}{x}, & 1 \\ e, & f, & 1-gx \end{vmatrix} = 0,$$

a quadratic for finding x; thence the ratios of x^2, xy, z, determining x, y, z individually.

(xxi.)
$$\left.\begin{aligned} ay+bz &= \frac{z}{x^2} \\ cz+dx &= \frac{z}{y^2} \\ ex+fy &= \frac{1}{z} \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} -\frac{yz}{xy}+bzx+axy &= 0 \\ cyz-\frac{zx}{xy}+dxy &= 0 \\ fyz+ezx-\frac{xy}{xy} &= 0 \end{aligned}\right\}.$$

The determinant gives xy; thence follow the ratios of xy, yz, zx.

(xxii.)
$$\left.\begin{aligned} ay+bz &= y^2z \\ cz+dx &= x^2z \\ ex+fy &= \frac{x^2y^2}{z} \end{aligned}\right\} \quad \text{or} \quad \left.\begin{array}{l} -xy\,.\,yz+bzx+axy = 0 \\ cyz-xy\,.\,zx+dxy = 0 \\ fyz+ezx-xy\,.\,xy = 0 \end{array}\right\}.$$

Here we find first xy, then the ratios of xy, yz, zx, whence x y, and z absolutely.

VIII. We will give three examples of equations which, though not in the first instance capable of solution by the method, can be so solved by transformation. The first is taken from a Paper in the Mathematical Tripos of 1860, the second from a Gonville and Caius Entrance Scholarship Paper of 1881, the third was proposed by Mr. Tucker in the *Educational Times* as Question 4309 (Vol. XXI., p. 40).

(xxiii.) $$x^2-yz = a^2,\quad y^2-zx = b^2,\quad z^2-xy = c^2.$$

Multiplying the equations first by x, y, z; then by z, x, y; then by y, z, x; and in each case adding, we obtain

$$\begin{aligned} a^2x+b^2y+c^2z &= (a^4-b^2c^2)\,x^{-1}, \\ b^2x+c^2y+a^2z &= 0, \\ c^2x+a^2y+b^2z &= 0, \end{aligned}$$

therefore
$$\begin{vmatrix} a^2-(a^4-b^2c^2)\,x^{-2}, & b^2, & c^2 \\ b^2, & c^2, & a^2 \\ c^2, & a^2, & b^2 \end{vmatrix} = 0,$$

giving
$$(a^4-b^2c^2)\,x^{-2} = a^6+b^6+c^6-3a^2b^2c^2,$$

and similarly for y and z

(xxiv.)
$$\left.\begin{aligned} y^2+z^2-x(y+z) &= a^2 \\ z^2+x^2-y(z+x) &= b^2 \\ x^2+y^2-z(x+y) &= c^2 \end{aligned}\right\}.$$

Multiplying and adding as in the previous example, we obtain

$$\left.\begin{aligned} c^2x + a^2y + b^2z &= x^3+y^3+z^3-3xyz \\ a^2x + b^2y + c^2z &= 0 \\ (b^2-c^2)\,x+(c^2-a^2)\,y+(a^2-b^2)\,z &= 0 \end{aligned}\right\} \text{............}(\alpha).$$

Put $(x^3+y^3+z^3-3xyz)/x = u$, then we have

$$\begin{vmatrix} c^2-u, & a^2, & b^2 \\ a^2, & b^2, & c^2 \\ b^2-c^2, & c^2-a^2, & a-b \end{vmatrix} = 0,$$

or
$$u\begin{vmatrix} b^2, & c^2 \\ c^2-a^2, & a^2-b^2 \end{vmatrix} = \begin{vmatrix} c^2, & a^2, & b^2 \\ a^2, & b^2, & c^2 \\ b^2-c^2, & c^2-a^2, & a^2-b^2 \end{vmatrix} = \begin{vmatrix} c^2, & a^2, & b^2 \\ a^2, & b^2, & c^2 \\ b^2, & c^2, & a^2 \end{vmatrix},$$

or
$$\frac{x^3+y^3+z^3-3xyz}{x} = \frac{a^6+b^6+c^6-3a^2b^2c^2}{b^4+c^4-a^2b^2-a^2c^2} \equiv k \text{ suppose.}$$

Substituting this in equations (α), we thereupon obtain the ratios of x, y, z in the form $y = mx,\ z = nx$; whence

$$\frac{x^2}{1} = \frac{y^2}{m^2} = \frac{z^2}{n^2} = \frac{k}{1+m^3+n^3-3mn}.$$

(xxv.)
$$\left.\begin{array}{r} -x^2+y^2+z^2-xy+yz-xz=a \\ x^2-y^2+z^2-xy-yz+xz=b \\ x^2+y^2-z^2+xy-yz-xz=c \end{array}\right\}.$$

Here we have
$$\begin{aligned} -ax-by-cz &= cx+ay+bz, \\ &= bx+cy+az, \\ &= x^3+y^3+z^3-3xyz, \end{aligned}$$

or
$$\left.\begin{array}{llll} \left(a+\dfrac{x^3+y^3+z^3-3xyz}{x}\right)x+ & by & + \quad cz & =0 \\ (a+c)\,x & +(b+a)\,y & +(c+b)\,z & =0 \\ (a+b)\,x & +(b+c)\,y & +(a+c)\,z & =0 \end{array}\right\},$$

the solution of which is effected as in the foregoing. Or we may, by subtraction, obtain

$$x^2-yz=\tfrac{1}{2}(b+c), \quad y^2-zx=\tfrac{1}{2}(c+a), \quad z^2-xy=\tfrac{1}{2}(a+b),$$

and complete the solution as in equation (xxiii.) on the preceding page. These may be compared with the more complicated methods of solving the question given on pp. 40, 41 of Vol. xxi. of the *Reprint*.

The above are sufficient illustrations of an apparently hitherto unnoticed application of determinants. It has not been thought necessary to give examples with numerical coefficients; nor cases where the determinant contains the function outside of the principal diagonal. The process is evidently capable of almost indefinite extension, and may be applied to equations containing any number of unknown quantities.

APPENDIX III.

SOLUTIONS OF SOME OLD QUESTIONS, BY ÂSÛTOSH MUKHOPÂDHYÂY, M.A., F.R.A.S.

1748. (By the late Professor CLIFFORD, F.R.S.)—Let X, Y, Z, U, V = 0 be the Cartesian equations, and r_1, r_2, r_3, r_4, r_5 the radii, of five spheres, cutting each other orthogonally; then identically

$$\frac{X^2}{r_1^2}+\frac{Y^2}{r_2^2}+\frac{Z^2}{r_3^2}+\frac{U^2}{r_4}+\frac{V^2}{r_5^2}=0,\quad \frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}+\frac{1}{r_5^2}=0.$$

Solution.

The results in this question are immediate consequences of the theory of power-coordinates; the subject has been considered by Professor CLIFFORD in two unfinished fragments, which are to be found in his *Mathematical Papers*, p. 332 and p. 546.

The equation of a sphere contains four available constants; the equation of *any* sphere may, therefore, be put into the form

$$aX+bY+cZ+dU+eV=0 \quad\text{..........................(1)},$$

wherein X, Y, Z, U, V = 0 are the equations of five given spheres; and any linear equation in X, Y, Z, U, V represents a sphere. To see the geometrical meaning of (1), let us write the values of X, Y, Z, U, V in their developed forms, viz.,

$$\left.\begin{aligned} X &= (x-\alpha_1)^2+(y-\beta_1)^2+(z-\gamma_1)^2-r_1^2 \\ Y &= (x-\alpha_2)^2+(y-\beta_2)^2+(z-\gamma_2)^2-r_2^2 \\ Z &= (x-\alpha_3)^2+(y-\beta_3)^2+(z-\gamma_3)^2-r_3^2 \\ U &= (x-\alpha_4)^2+(y-\beta_4)^2+(z-\gamma_4)^2-r_4^2 \\ V &= (x-\alpha_5)^2+(y-\beta_5)^2+(z-\gamma_5)^2-r_5^2 \end{aligned}\right\}\text{....................(2)},$$

which show that X means the squared tangent from the point (x, y, z) to the first sphere, or it is the power of that point with respect to the sphere; and hence the geometrical meaning of (1) is, that if the squared tangents drawn from a point to five fixed spheres satisfy a linear equation, the locus of that point is a sphere. Therefore X, Y, Z, U, V may be regarded as furnishing a sort of coordinates; and, as (1) is homogeneous, we may define the coordinates of a sphere to be quantities *proportional* to certain multiples of its powers in respect of five fixed spheres, not having the same radical centre. The expressions occurring in the theory are

much simplified by taking as multiples the reciprocals of the radii of the fixed spheres.

Now, if we have two sets of six spheres A, B, C, D, E, F and P, Q, R, S, T, W; and if we form the determinant whose constituents are the powers of the first set of spheres in respect of the second set, so as to be represented in the umbral notation by

$$\begin{pmatrix} A & B & C & D & E & F \\ P & Q & R & S & T & W \end{pmatrix},$$

this determinant, by an easy application of the theory of matrices, is seen to vanish identically. As a particular case, let us take, for the first five spheres of each set, the fundamental spheres X, Y, Z, U, V, and consider two other spheres, 6, 6′. Then, identically,

$$\begin{pmatrix} X, & Y, & Z, & U, & V, & 6 \\ X, & Y, & Z, & U, & V, & 6' \end{pmatrix} \equiv 0.$$

Then, if the coordinates of 6, 6′ be represented by x, y, z, u, v, and x', y', z', u', v', respectively, by expanding the above identity, we have the relation

$$\begin{vmatrix} -2r_1^2 & (XY) & (XZ) & (XU) & (XV) & x \\ (XY) & -2r_2^2 & (YZ) & (YU) & (YV) & y \\ (XZ) & (YZ) & -2r_3^2 & (ZU) & (ZV) & z \\ (XU) & (YU) & (ZU) & -2r_4^2 & (UV) & u \\ (XV) & (YV) & (ZV) & (UV) & -2r_5^2 & v \\ x' & y' & z' & u' & v' & (6, 6') \end{vmatrix} \equiv 0.$$

Now, suppose 6, 6′ to coincide in a sphere of radius r; then, dividing the first five rows and columns by r_1, r_2, r_3, r_4, r_5 respectively, and putting $r = 0$, we have

$$\begin{vmatrix} 1 & \cos XY & \cos XZ & \cos XU & \cos XW & \frac{X}{r_1} \\ \cos XY & 1 & \cos YZ & \cos YU & \cos YW & \frac{Y}{r_2} \\ \cos XZ & \cos YZ & 1 & \cos ZU & \cos ZW & \frac{Z}{r_3} \\ \cos XU & \cos YU & \cos ZU & 1 & \cos UW & \frac{U}{r_4} \\ \cos XW & \cos YW & \cos ZW & \cos UW & 1 & \frac{V}{r_5} \\ \frac{X}{r_1} & \frac{Y}{r_2} & \frac{Z}{r_3} & \frac{U}{r_4} & \frac{V}{r_5} & 0 \end{vmatrix} \equiv 0.$$

In the theory of power-coordinates, this is to be viewed as a relation to be satisfied in order that the radius of the sixth sphere may vanish; but, in the ordinary Cartesian theory, this is an *identical* relation satisfied by the equations of any five spheres. In the particular case, when the five spheres form a mutually orthogonal system, the elements involving the cosine-function in the above determinant clearly vanish; and then the relation reduces to $\frac{X^2}{r_1^2} + \frac{Y^2}{r_2^2} + \frac{Z^2}{r_3^2} + \frac{U^2}{r_4^2} + \frac{V^2}{r_5^2} = 0,$

which is exactly the first theorem in question.

The second theorem is proved in exactly a similar manner, viz., here, instead of supposing the radius of the sixth sphere to be evanescent, we suppose it to become infinite, so that the sphere, instead of degenerating to a point, enlarges into a plane. Then, dividing the first five rows and columns of the determinant

$$\begin{pmatrix} X, & Y, & Z, & U, & V, & 6 \\ X, & Y, & Z, & U, & V, & 6' \end{pmatrix} \equiv 0$$

by Xr_1, Yr_2, Zr_3, Ur_4, Vr_5, and noticing that in this case the powers of the plane with respect to the spheres are infinite, so that their reciprocals vanish, we get the theorem in question. This also follows in a slightly different way; viz., we know that, when the fundamental spheres form an orthogonal system, the Second Absolute becomes

$$\phi = \frac{X^2}{r_1^2} + \frac{Y^2}{r_2^2} + \frac{Z^2}{r_3^2} + \frac{U^2}{r_4^2} + \frac{V^2}{r_5^2},$$

and the radius of (x, y, z, u, v) is given by

$$4\rho = \frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} + \frac{z^2}{r_3^2} + \frac{u^2}{r_4^2} + \frac{v^2}{r_5}.$$

We further know that a point or sphere of zero radius satisfies the analytical condition of touching the Second Absolute, while a plane or sphere of infinite radius has its powers infinite. Thus, by putting $\phi = 0$,

we have $$\frac{X^2}{r_1^2} + \frac{Y^2}{r_2^2} + \frac{Z^2}{r_3^2} + \frac{U^2}{r_4^2} + \frac{V^2}{r_5^2} = 0,$$

and, by putting x, y, z, u, v each infinite, we have

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} + \frac{1}{r_5^2} = 0.$$

It is to be distinctly remembered that, from the point of view of Cartesian coordinates, these are *identical* relations connecting the equations and radii of five mutually orthotomic spheres; while, from the point of view of power-coordinates, they are certain homogeneous relations that must be satisfied in order that the sixth sphere may contract into a point or expand out to a plane, respectively.

3135. (By R. Tucker, M.A.)—If the sides of a plane triangle in order of magnitude be a, b, c, show that it is *not always* possible to form a triangle with the escribed radii. Given b, c, find the maximum triangle that can be formed when a varies.

Solution.

Let a, b, c be the sides, and Δ the area of the given triangle; let α, β, γ be the sides, and P the area of the new triangle formed with the ex-radii. Then, we have

$$\alpha = r_a = \frac{\Delta}{s-a}, \quad \beta = r_b = \frac{\Delta}{s-b}, \quad \gamma = r_c = \frac{\Delta}{s-c} \ldots\ldots (1, 2, 3),$$

where, as usual, $$2s = a + b + c.$$

$$16\Delta^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$16P^2 = 2\beta^2\gamma^2 + 2\gamma^2\alpha^2 + 2\alpha^2\beta^2 - \alpha^4 - \beta^4 - \gamma^4 \ldots\ldots\ldots\ldots\ldots (5).$$

Now, the three conditions which must be satisfied in order that a triangle may be formed with the ex-radii, are

$$\alpha+\beta>\gamma, \quad \beta+\gamma>\alpha, \quad \gamma+\alpha>\beta.$$

Taking the first of these, we must have

$$\frac{1}{s-a}+\frac{1}{s-b}>\frac{1}{s-c}, \quad \text{or} \quad \frac{c}{s^2-s(a+b)+ab}>\frac{1}{s-c},$$

whence $$s^2>c^2+ab.$$

Similarly, attending to the other two conditions, we infer that a triangle may be constructed with the ex-radii only when s^2 is greater than each of the three quantities a^2+bc, b^2+ca, c^2+ab.

When a varies, it is clear, from an inspection of the first five equations that α, β, γ, s, Δ, P are all functions of a, while b and c are constants. Therefore, differentiating (5) with regard to a, we get

$$32\mathrm{P}\frac{d\mathrm{P}}{da}=4\alpha\frac{d\alpha}{da}(\beta^2+\gamma^2-\alpha^2)+4\beta\frac{d\beta}{da}(\gamma^2+\alpha^2-\beta^2)+4\gamma\frac{d\gamma}{da}(\alpha^2+\beta^2-\gamma^2).$$

Hence, when P is a maximum, as $\frac{d\mathrm{P}}{da}=0$, we must have

$$\alpha\frac{d\alpha}{da}(\beta^2+\gamma^2-\alpha^2)+\beta\frac{d\beta}{da}(\gamma^2+\alpha^2-\beta^2)+\gamma\frac{d\gamma}{da}(\alpha^2+\beta^2-\gamma^2)=0 \quad \ldots(6).$$

Again, differentiating the first three equations, we obtain

$$\frac{d\alpha}{da}=\frac{(s-a)\frac{d\Delta}{da}-\Delta\left(\frac{ds}{da}-1\right)}{(s-a)^2}, \quad \frac{d\beta}{da}=\frac{(s-b)\frac{d\Delta}{da}-\Delta\frac{ds}{da}}{(s-b)^2} \quad \ldots\ldots(7,\ 8),$$

$$\frac{d\gamma}{da}=\frac{(s-c)\frac{d\Delta}{da}-\Delta\frac{ds}{da}}{(s-c)^2}\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Moreover, differentiating (4), as well as the equation, $2s=a+b+c$, we get

$$32\Delta\frac{d\Delta}{da}=4a(c^2+b^2-a^2),$$

whence $$8\Delta\frac{d\Delta}{da}=a(c^2+b^2-a^2)=8\Delta\mathrm{R}, \text{ say, and } \frac{ds}{da}=\tfrac{1}{2} \ldots(10,\ 11).$$

Substituting from (10) and (11) in (7), (8), (9), we obtain

$$(s-a)^2 d\alpha=\mathrm{R}(s-a)+\tfrac{1}{2}\Delta, \quad (s-b)^2\frac{d\beta}{da}=\mathrm{R}(s-b)-\tfrac{1}{2}\Delta \ldots(12,\ 13),$$

$$(s-c)^2\frac{d\gamma}{da}=\mathrm{R}(s-c)-\tfrac{1}{2}\Delta \quad \ldots\ldots\ldots\ldots\ldots(14).$$

Now, substituting from (1), (2), (3) in (6), we get

$$\frac{1}{s-a}\left\{\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}-\frac{1}{(s-a)^2}\right\}\frac{d\alpha}{da}+\frac{1}{s-b}\left\{\frac{1}{(s-c)^2}+\frac{1}{(s-a)^2}-\frac{1}{(s-b)^2}\right\}\frac{d\beta}{da}$$
$$+\frac{1}{s-c}\left\{\frac{1}{(s-a)^2}+\frac{1}{(s-b)}-\frac{1}{(s-c)^2}\right\}\frac{d\gamma}{da}=0 \quad \ldots\ldots\ldots(15).$$

Next, put $s-a=\lambda^{-1}$, $s-b=\mu^{-1}$, $s-c=\nu^{-1}$, and substitute from (12), (13), (14) in (15), so that

$$\mathrm{R}(2\lambda^2\mu^2+2\mu^2\nu^2+2\nu^2\lambda^2-\lambda^4-\mu^4-\nu^4)$$
$$+\tfrac{1}{2}\Delta\left[\lambda^3(\mu^2+\nu^2-\lambda^2)-\mu^3(\nu^2+\lambda^2-\mu^2)-\nu^3(\lambda^2+\mu^2-\nu^2)\right]=0,$$

which is the final equation from which a has to be determined in terms of b, c; this equation is so complicated that I do not suppose it would be at all practicable to determine actually the area of the maximum triangle. It is worth noting that a beautifully symmetrical result is immediately derivable from (6), viz., if θ, ϕ, ψ be the angles of the triangle formed with the ex-radii, we have

$$\beta^2+\gamma^2-\alpha^2 = 2\beta\gamma\cos\theta,$$

with two other similar relations, whence at once

$$\cos\theta\frac{d\alpha}{d\alpha}+\cos\phi\frac{d\beta}{d\alpha}+\cos\psi\frac{d\gamma}{d\alpha}=0,$$

or, not restricting the independent variable to α, this becomes

$$\cos\theta\,.\,d\alpha+\cos\phi\,.\,d\beta+\cos\psi\,.\,d\gamma=0.$$

3189. (By Professor EVANS, M.A.)—Prove that the product of the six consecutive numbers

$$P=(x-5)(x-4)(x-3)(x-2)(x-1)x$$

cannot be a perfect square for any integral or commensurable value of x.

Solution.

We have

$$\begin{aligned} P &= (x^2-5x)(x^2-5x+4)(x^2-5x+6) \\ &= (x^2-5x)^3+10(x^2-5x)^2+24(x^2-5x) \\ &= x^6-15x^5+85x^4-225x^3+274x^2-120x \\ &= \left(x^3-\tfrac{15}{2}x^2+\tfrac{115}{8}x-\tfrac{75}{16}\right)^2-\tfrac{9}{64}\left(21x^2+105x-\tfrac{625}{4}\right). \end{aligned}$$

Therefore P is a perfect square, when

$$21x^2+105x=\tfrac{625}{4},\quad \text{which gives}\quad x=-\tfrac{5}{2}\pm5\sqrt{\tfrac{23}{42}},$$

from the very form of which the truth of the theorem in question is manifest.

[Another solution is given on p. 65 of this Volume.]

3193. (By J. H. TURRELL, M.A.)—Find a point such that the sum of the cubes of the perpendiculars drawn from it on the sides of a triangle shall be equal to the area.

Solution.

Let α, β, γ be the trilinear coordinates of the point; a, b, c the sides, and Δ the area of the given triangle, which may be taken as the triangle of reference. Then $\alpha^3+\beta^3+\gamma^3=\Delta$.

Making this homogeneous by means of the identical relation

$$a\alpha+b\beta+c\gamma=2\Delta,$$

we have

$$8\Delta^2(\alpha^3+\beta^3+\gamma^3)=(a\alpha+b\beta+c\gamma)^3,$$

which shows that the position of the point is *not* unique, but that every point on this cubic curve satisfies the condition of the question. Writing the equation in its developed form, we find it to be

$$\begin{aligned}\bar{a}\beta\gamma + \alpha^2 &\left[\tfrac{1}{8}(a^3-8\Delta^2)\,\alpha + \tfrac{1}{2}a^2b\,.\,\beta + \tfrac{1}{2}a^2c\,.\,\gamma\right] \\ + \beta^2 &\left[\tfrac{1}{2}ab^2\,.\,\alpha + \tfrac{1}{8}(b^3-8\Delta^2)\,\beta + \tfrac{1}{2}b^2c\,.\,\gamma\right] \\ + \gamma^2 &\left[\tfrac{1}{2}ac^2\,\alpha + \tfrac{1}{2}bc^2\beta + \tfrac{1}{8}(c^3-8\Delta^2)\,\gamma\right] = 0,\end{aligned}$$

whence we at once infer that the *only* case in which this cubic passes through the angular points of the triangle of reference is when

$$a^3 = b^3 = c^3 = 8\Delta^2,$$

which requires it to be an equilateral triangle whose side is $\frac{2}{3}$, and the equation of the cubic then reduces to

$$27\,\alpha\beta\gamma + 4\alpha^2(\beta+\gamma) + 4\beta^2(\gamma+\alpha) + 4\gamma^2(\alpha+\beta) = 0,$$

or

$$4(\alpha+\beta+\gamma)^3 + 57\,\alpha\beta\gamma = 4(\alpha^3+\beta^3+\gamma^3).$$

3194. (By the late G. O. HANLON.)—Find the envelope of a straight line which is cut harmonically by two given conics; and determine the conditions under which the envelope becomes a conic.

Solution.

Let $(a_1, b_1, c_1, f_1, g_1, h_1)(\alpha, \beta, \gamma) = 0$, $(a_2, b_2, c_2, f_2, g_2, h_2)(\alpha, \beta, \gamma) = 0$ be the equations of the two conics, where

$$(a, b, c, f, g, h)(\alpha, \beta, \gamma) \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0.$$

The points of intersection of the first conic with $\lambda\alpha + \mu\beta + \nu\gamma = 0$, which may be taken to be the equation to the given line, are found, by eliminating γ between the two equations, to satisfy

$$\begin{aligned}(c_1\lambda^2 + a_1\nu^2 - 2g_1\lambda\nu)\,\alpha^2 + 2\,(c_1\lambda\mu - f_1\lambda\nu - g_1\mu\nu + h_1\nu^2)\,\alpha\beta \\ + (c_1\mu^2 + b_1\nu^2 - 2f_1\mu\nu)\,\beta^2 = 0.\end{aligned}$$

The points of intersection of the second conic with the same line satisfy a similar equation obtainable from this one by changing the suffix 1 into 2. Applying, then, the usual condition that these four points may be harmonically conjugate, the required condition is found to be

$$A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0,$$

where A, B, C, F, G, H are certain well-known constants, viz.,

$$\begin{array}{l|l} A = b_1c_2 + b_2c_1 - 2f_1f_2 & F = g_1h_2 + g_2h_1 - a_1f_2 - a_2f_1, \\ B = c_1a_2 + c_2a_1 - 2g_1g_2 & G = h_1f_2 + h_2f_1 - b_1g_2 - b_2g_1, \\ C = a_1b_2 + a_2b_1 - 2h_1h_2 & H = f_1g_2 + f_2g_1 - c_1h_2 - c_2h_1. \end{array}$$

Hence, finally, we have to find the envelope of $\lambda\alpha + \mu\beta + \nu\gamma = 0$, when λ, μ, ν are connected by a quadric relation, and, therefore, the required envelope is *always* a line of the second order.

As an illustration of the general method, let us obtain the envelope of $\lambda x + \mu y + \nu z = 0$, when it is cut harmonically by the conics

$$x^2 + y^2 + z^2 = 0, \quad a^2x^2 + b^2y^2 + c^2z^2 = 0 \;\ldots\ldots\ldots\ldots\ldots(1, 2).$$

The results of eliminating z between the equation to the line and the equations of the conics, are found to be

$$(\lambda^2+\nu^2)(x/y)^2 + 2\lambda\mu\,(x/y) + (\mu^2+\nu^2) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$(c^2\lambda^2 + a^2\nu^2)(x/y)^2 + 2c^2\lambda\mu\,(x/y) + (c^2\mu^2 + b^2\nu^2) = 0 \ldots\ldots\ldots\ldots(4).$$

The condition that the points given by (3) and (4) may be harmonically conjugate is $(\lambda^2+\nu^2)(c^2\mu^2+b^2\nu^2) + (\mu^2+\nu^2)(c^2\lambda^2+a^2\nu^2) = 2c^2\lambda^2\mu^2$, which reduces to

$$(b^2+c^2)\,\lambda^2 + (c^2+a^2)\,\mu^2 + (a^2+b^2)\,\nu^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Now, the envelope of $\lambda x + \mu y + \nu z = 0$, when the parameters satisfy (5), is easily found to be $\dfrac{x^2}{b^2+c^2} + \dfrac{y^2}{c^2+a^2} + \dfrac{z^2}{a^2+b^2} = 0$; which is, accordingly, the equation to the enveloping conic.

3195. (By J. B. SANDERS.)—Find the least velocity with which a body must be projected from the Moon, in the direction of a line joining the centre of the Earth and Moon, so that it may just reach the Earth.

Solution.

Consider the Earth and Moon to be homogeneous material spheres; let μ_1 be the mass, and, r_1 the radius of the Moon; μ_2, r_2 the corresponding elements for the Earth; a, the distance apart between the centres of the two bodies; V, the velocity with which the body is projected from the surface of the Moon, in the direction of the centre-line, which may be taken as the axis of x, emanating from the centre of the Moon as origin. Then, the equation of motion is

$$\frac{d^2x}{dt^2} = -\left\{\frac{\mu_2}{(a-x)^2} - \frac{\mu_1}{x^2}\right\}.$$

Integrating, $$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 = -\frac{\mu_1}{x} - \frac{\mu_2}{a-x} + \text{C},$$

where C is the constant of integration.

Now, when $x = r_1$, $\dfrac{dx}{dt} = \text{V}$, and when $x = a - r_2$, $\dfrac{dx}{dt} = 0$,

which give $$\tfrac{1}{2}\text{V}^2 = -\frac{1}{r_1} - \frac{\mu_2}{a-r_1} + \text{C}, \quad 0 = -\frac{\mu_1}{a-r_2} - \frac{2}{a-r_2} + \text{C},$$

therefore $$\tfrac{1}{2}\text{V}^2 = \mu_1\left(\frac{1}{a-r_2} - \frac{1}{r_1}\right) + \mu_2\left(\frac{1}{r_2} - \frac{1}{a-r_1}\right)$$

$$= \{a-(r_1+r_2)\}\left\{-\frac{\mu_1}{r_1(a-r_2)} + \frac{\mu_2}{r_2(a-r_1)}\right\},$$

which gives the value of V in terms of known quantities.

An approximate numerical solution may be obtained by taking the rough values, $a = 237640$, $r_1 = 1077$, $r_2 = 3963$, $\mu_1 = 1$, $\mu_2 = 80$. These give

$$\tfrac{1}{2}\text{V}^2 = 232600\left(\frac{80}{3963 \times 236563} - \frac{1}{1077 \times 233677}\right)$$

$$\text{V}^2 = \frac{232600 \times 19196111151 \times 2}{1077 \times 3963 \times 233677 \times 236563}$$

$$= \cdot 03836, \text{ approximately.}$$

3196. (By Professor HUDSON, M.A.)—A heavy string hangs from two points in the same horizontal plane. If the curve in which it hangs, referred to a horizontal tangent be $s = a\tan^2\phi$, prove that the density of the string at any point varies as the tangent of the inclination to the vertical.

Solution.

Let the origin be a point on the curve, and the axis of x the horizontal tangent at that point. Then, since

$$\frac{dy}{dx} = \tan\phi, \quad s = a\tan^2\phi,$$

we have $$\frac{dy}{dx} = a^{-\frac{1}{2}}s^{\frac{1}{2}}, \quad \frac{d^2y}{dx^2} = \tfrac{1}{2}a^{-\frac{1}{2}}s^{\frac{1}{2}}\frac{ds}{dx} = \tfrac{1}{2}a^{-1}(\tan\phi)^{-1}\frac{ds}{dx},$$

whence we have $$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{\cot\phi}{2a}.$$

Now from the ordinary theory of the equilibrium of flexible inextensible strings, we know that $\frac{d}{ds}\left(\frac{dy}{dx}\right)$ varies as the density, therefore the density varies as $\cot\phi$, that is, as the tangent of the angle of inclination to the vertical. The inverse problem, viz., to find the catenary which possesses this property, also comes out very neatly; for if the density vary as $\cot\phi$, *i.e.*, as $\frac{dx}{dy}$, we have $\frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{2}{a}\frac{dx}{dy}$; therefore $\frac{a}{2}\frac{dy}{dx}\frac{d}{ds}\left(\frac{dy}{dx}\right) = 1$; or $a\left(\frac{dy}{dx}\right)^2 = s$, if we remove s from the horizontal tangent, which is $s = a\tan^2\phi$.

3240. (By Professor GENESE, M.A.)—Prove that, if the equation $x^3-3px+q=0$ has (i.) only one root ($=\alpha$) real, then $\alpha^2>4p$; and (ii.), if all the three roots be real, any one must be numerically less than $2p^{\frac{1}{2}}$.

Solution.

From the ordinary theory of the criterion of the nature of the roots of a cubic, we know that the cubic has only one root or all the three roots real, according as G^2+4H^3 is positive or negative. This test, as applied to the equation $$x^3-3px+q=0,$$ shows that only one root or all the three roots are real, according as q^2-4p^3 is positive or negative. Now, if α be a root of this equation, we have $\alpha^3-3p\alpha+q=0$, with the conditions

$$q^2 > \text{ or } < 4p^3, \quad \text{that is, } q > \text{ or } < 2p^{\frac{3}{2}},$$

or $$3p\alpha-\alpha^3 > \text{ or } < 2p^{\frac{3}{2}}, \quad \text{or} \quad 2p^{\frac{3}{2}}-3p\alpha+\alpha^3 < \text{ or } > 0,$$

or $$(2p^{\frac{1}{2}}+\alpha)\left[p-\alpha\,(2p^{\frac{1}{2}}-\alpha)\right] < \text{ or } > 0.$$

Considering only the first factor, we have $4p < \text{or} > \alpha^2$, which proves both parts of the question.

[*Otherwise* :—Let $x=\alpha$ be a root of $x^3-3px+q=0$; then $(x-\alpha)$ divides the "sinister," and for the other roots we have $x^2+\alpha x+\alpha^2-3p=0$. These are real, or imaginary according as

$$\alpha^2 > \text{ or } < 4(\alpha^2-3p), \; i.e., \text{ as } 12p > \text{ or } < 3\alpha \text{ or as } \alpha^2 < \text{ or } > 4p.$$

The proof in the text does not apply if α be imaginary. Professor GENESE states that the question was "a happy thought" inspired by horror of the approximations in TODHUNTER's *Theory of Equations*.]

3270 & 5378. (By Professor HUDSON, M.A.)—A cylinder is filled with incompressible fluid, the density of which varies as the square of the depth. Find the generating curve of a solid of revolution, into which it must be poured, in order that the density, when it settles, may vary as the fourth power of the depth.

Solution.

Measure x upwards from the lowest point of the liquid; let a be the

radius of the cylinder, and h, h_1 the depths of the liquid in the two vessels. Then, if ρ be the density, we have

$$\rho = \mu(h-x)^2 = \mu_1(h_1-x_1)^4,$$

which gives $$x = h - \frac{1}{c}(h_1-x_1)^2, \quad \text{if} \quad \mu = \mu_1 c^2.$$

Also, equating the volumes of corresponding strata, we get

$$\pi a^2 . dx = \pi y_1^2 dx_1,$$

and, as $dx = \frac{2}{c}(h_1-x_1)dx_1$, this gives

$$\frac{2a^2}{c}(h_1-x_1) = y_1^2.$$

To determine the constant c, we equate the total volumes, that is,

$$\int_0^h \pi a^2 . dx = \int_0^{h_1} \frac{2\pi a^2}{c}(h_1-x_1)\,dx_1,$$

which gives $h_1^2 = ch$. Therefore the final equation is

$$y_1^2 = \frac{2ha^2}{h_1^2}(h_1-x_1),$$

which shows that the generating curve of the required surface of revolution is a parabola.

3308 & 5274. (By the late Professor CLIFFORD, F.R.S.)—Prove that the forty umbilici of an anallagmatic surface lie by fives on sixteen straight lines.

Solution.

Sir WILLIAM HAMILTON has remarked (*Elements of Quaternions*, p. 662) that a central quadric has, in general, twelve umbilics, whereof only four at most can be real, and which are its intersections with the three focal curves; and these twelve points are ranged, three by three, on eight imaginary right lines, which intersect the circle at infinity, and which are called the eight umbilicar generatrices of the surface. The present theorem of CLIFFORD's may be regarded as an extension of HAMILTON's theorem to quartics.

An anallagmatic quartic has, for the nodal curve, the circle at infinity; and it has been shown by CLEBSCH (SALMON's *Surfaces*, § 586) that, on the surface, there are sixteen right lines which intersect the imaginary circle, and each of which is met by five others, thus giving rise to $\frac{1}{2} \times 5 \times 16 = 40$ umbilical intersections; and, as at an umbilic the inflexional tangents are lines of no length, they lie entirely on the surface, which proves exactly the theorem in question. As remarked by SALMON (§ 559), the theorem may be shown to be true for cubics, by the method of inversion.

3351. (By J. B. SANDERS.)—A body is projected at a given distance from the centre of force, with a given velocity, and in a direction perpendicular to the distance, when the force is repulsive and varies inversely as the cube of the distance. Find the path of the body.

Solution.

We have, generally, $\qquad \dfrac{d^2u}{d\theta^2} + u = \dfrac{P}{h^2u^2},$

where P is the central force. If $P = -\mu u^3$, the equation of motion is

$$\frac{d^2u}{d\theta^2} + u + \frac{\mu}{h^2}\,u = 0.$$

If $1 + \dfrac{\mu}{h^2} = k^2$, we get $\qquad \dfrac{d^2u}{d\theta^2} + k^2u = 0,$

the integral of which is $\qquad au = \cos k(\theta - \alpha),$

whence $\qquad a\dfrac{du}{d\theta} = -k \sin k(\theta - \alpha).$

Then α is the value of θ, corresponding to an apse, and a is the apsidal distance.

3353. (By T. Mitcheson, B.A.)—A long stick rests on the edge and bottom of a vessel filled with water; and the submerged part, $(8\frac{566}{983})^{\frac{1}{2}}$ feet long, appears to make with the portion in air an angle of 150°. Find the perpendicular depth of the vessel.

Solution.

Let x be the perpendicular depth of the vessel, and l the length of the portion of the stick submerged. Then $x = l\cos\phi'$. Since the angle between the incident and refracted rays is 150°, we have

$$\phi' + 90^\circ + 90^\circ - \phi = 150^\circ, \quad \text{whence} \quad \phi = \phi' + \tfrac{1}{6}\pi,$$

which gives $\qquad \mu = \dfrac{\sin\phi}{\sin\phi'} = \dfrac{\sin(\phi' + \frac{1}{6}\pi)}{\sin\phi'} = \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\cot\phi',$

or $\qquad \cot\phi' = 2\mu - \sqrt{3},$

whence $\qquad \cos\phi' = \dfrac{1}{2\left[1 - \sqrt{3}\,.\,\mu + \mu^2\right]^{\frac{1}{2}}},$

therefore $\qquad x = \dfrac{l}{2\left[1 - \sqrt{3}\,.\,\mu + \mu^2\right]^{\frac{1}{2}}};$

from which the depth is known, since l is given, and μ can be determined experimentally, viz., $\mu = 1{\cdot}33 = \frac{4}{3}$ approximately. [See Jamin et Bouty, t. iii., p. 62.]

3364 & 5127. (By the Editor.)—A curve is described by setting off, on the focal radius vectors of an ellipse, distances inversely proportional to them, and having equal greatest and least values. Find the ratio of the mean of all the radii vectores to the mean of all the radii vectores of the ellipse, and the ratio of their areas.

Solution.

Let the ellipse be $\qquad \dfrac{l}{r} = 1 + e\cos\theta.$

When $\theta = 0$, $\quad r = \dfrac{l}{1+e}$, a minimum,

$\theta = \pi$, $\quad r = \dfrac{l}{1-e}$, a maximum.

Putting $\dfrac{\lambda}{r}$ for r, where λ is a constant, we get for the equation of the derived curve
$$\frac{lr}{\lambda} = 1 + e\cos\theta.$$

When $\theta = 0$, $\quad r = \dfrac{\lambda(1+e)}{l}$, a maximum,

$\theta = \pi$, $\quad r = \dfrac{\lambda(1-e)}{l}$, a minimum.

The additional condition, that the maximum and minimum values of the radii vectores of the primitive and derived curves are equal, leads to the relation $\quad \dfrac{l}{1+e} = \dfrac{\lambda(1-e)}{l}$, which gives $\lambda = \dfrac{l^2}{1-e^2}$,

so that the equation to the derived curve becomes
$$\frac{(1-e^2)\,r}{l} = 1 + e\cos\theta.$$

Let M_1, M_2 be the mean values of all the radii vectores of the two curves; so that
$$M_1 = \int_0^\pi \frac{l\,d\theta}{1+e\cos\theta} \bigg/ \int_0^\pi d\theta = \frac{\pi l}{(1-e^2)^{\frac{1}{2}}}\cdot\frac{1}{\pi} = \frac{l}{(1-e^2)^{\frac{1}{2}}},$$
$$M_2 = \int_0^\pi \frac{l}{1-e^2}(1+e\cos\theta)\,d\theta \bigg/ \int_0^\pi d\theta = \frac{\pi l}{1-e^2}\cdot\frac{1}{\pi} = \frac{l}{1-e^2};$$
therefore
$$\frac{M_1}{M_2} = (1-e^2)^{\frac{1}{2}} = \frac{b}{a},$$
where a, b are the semi-axes of the ellipse.

Again, let A, B be the areas of the two curves. Then,
$$A = \int_0^\pi \frac{l^2\,d\theta}{(1+e\cos\theta)^2} = \frac{\pi l^2}{(1-e^2)^{\frac{3}{2}}},$$
$$B = \int_0^\pi \frac{l^2}{(1-e^2)^2}(1+e\cos\theta)^2\,d\theta = \frac{\pi l^2(2+e^2)}{2(1-e^2)^2}.$$
Therefore
$$\frac{A}{B} = 2\,\frac{(1-e^2)^{\frac{1}{2}}}{2+e^2} = \frac{2ab}{3a^2-b^2}.$$

3371. (By J. B. Sanders.)—Required the velocity and time when a material particle is attracted to a fixed point by a force varying inversely as the square root of the distance.

Solution.

The equation of motion is $\quad \dfrac{d^2x}{dt^2} = -\dfrac{\mu}{x^{\frac{1}{2}}}$ ....(1).

Multiplying by $2\dfrac{dx}{dt}$, and integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = 4\mu\,(a - x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

supposing the particle to start from rest at a distance a from the fixed point. Hence, if v be the velocity at any distance x from the centre of force, we have

$$v = 2\mu^{\frac{1}{2}}\,(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

To obtain the time, we have, from (2),

$$2\mu^{\frac{1}{2}}\,dt = \frac{dx}{(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{1}{2}}}.$$

Hence, if T be the time of falling to the centre from the position of rest, we have

$$2\mu^{\frac{1}{2}}\,\mathrm{T} = \int_0^a \frac{dx}{(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{1}{2}}}.$$

By putting $x = a\sin^4\theta$, we get

$$2\mu^{\frac{1}{2}}\,\mathrm{T} = 4a^{\frac{3}{4}}\int_0^{\frac{1}{2}\pi} \sin^3\theta\,d\theta = \tfrac{8}{3}a^{\frac{3}{4}},$$

whence

$$\mathrm{T} = \tfrac{4}{3}a^{\frac{3}{4}}\,\mu^{-\frac{1}{2}}.$$

The more general case, viz., when the action of attraction varies inversely as the n^{th} power of the distance, *n being less than unity*, is easily treated in the same manner; viz., the equation of motion being

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^n},$$

we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{1-n}\,(a^{1-n} - x^{1-n}),$$

where, as before, a is the distance from which the particle starts at rest. Therefore the velocity is given by the formula

$$v = \left(\frac{2\mu}{1-n}\right)^{\frac{1}{2}}(a^{1-n} - x^{1-n})^{\frac{1}{2}}.$$

To find the time, we have $\left(\dfrac{2\mu}{1-n}\right)^{\frac{1}{2}}dt = \dfrac{dx}{(a^{1-n} - x^{1-n})^{\frac{1}{2}}}.$

If T be the time required, this gives

$$\left(\frac{2\mu}{1-n}\right)^{\frac{1}{2}}\mathrm{T} = \int_0^a \frac{dx}{(a^{1-n} - x^{1-n})^{\frac{1}{2}}}.$$

By substituting $x = a\,(\sin\theta)^{\frac{2}{1-n}}$, we get

$$\left(\frac{2\mu}{1-n}\right)\ \mathrm{T} = \frac{2}{1-n}\,a^{\frac{1}{2}(1+n)}\int_0^{\frac{1}{2}\pi}(\sin\theta)^{\frac{1+n}{1-n}}\,d\theta$$

$$= \frac{2}{1-n}\,a^{\frac{1}{2}(1+n)}.\tfrac{1}{2}\sqrt{\pi}.\Gamma\left(\frac{1}{1-n}\right)\Big/\Gamma\left(\frac{1}{2}\cdot\frac{3-n}{1-n}\right);$$

from which we find

$$\mathrm{T} = \left\{\frac{1}{2}\cdot\frac{a^{1+n}}{1-n}\cdot\frac{\pi}{\mu}\right\}^{\frac{1}{2}}\Gamma\left(\frac{1}{1-n}\right)\Big/\Gamma\left(\frac{1}{2}\cdot\frac{3-n}{1-n}\right).$$

Expressions for the time and velocity, in the cases where n is either equal to or greater than unity, are well-known.

[See TAIT and STEELE's *Dynamics*, Arts. 101—104.]

3714 & 5185. (By Professor WHITWORTH, M.A.)—If two concentric and coaxial ellipses be equal in area, the eccentric angles at a common point will be complementary. Consider the limiting case when the ellipses approach similitude, and determine the locus of ultimate intersections of a series of such ellipses.

Solution.

Let one of the ellipses be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(1).

Then the equation of any other ellipse of equal area ($= \pi ab$), and concentric and coaxial with this one, may obviously be written

$$\frac{x^2}{\lambda^2 a^2} + \frac{\lambda^2 y^2}{b^2} = 1 \qquad \text{.................................(2).}$$

Then any point (ξ, η) on (1) is $\xi = a\cos\phi$, $\eta = b\sin\phi$, where, as usual, ϕ is the eccentric angle. If this point be also on (2), and θ be the eccentric angle with regard to (2), we have

$$\xi = a\lambda\cos\theta, \quad \eta = \frac{b}{\lambda}\sin\theta;$$

whence $\quad \cos\phi = \lambda\cos\theta, \quad \lambda\sin\phi = \sin\theta, \quad$ or $\quad \sin 2\phi = \sin 2\theta$;

and, *as ϕ is not equal to θ* (which would make the ellipses coincident), we must have $2\phi = \pi - 2\theta$, or $\phi = \frac{1}{2}\pi - \theta$, which shows that the eccentric angles at a common point are complementary.

If the ellipses are similar as well as similarly placed, we may have

$$\frac{a^2}{a^2\lambda^2} = \frac{b^2\lambda^2}{b^2},$$

which requires $\lambda = 1$, and this solution is extraneous, as requiring the ellipses to be coincident. The only *real* solution is that which requires the major axis of the one to be at right angles to that of the other, viz., this is so when $\quad \frac{a^2\lambda^2}{b^2} = \frac{b^2}{a^2\lambda^2}$,

whence $\quad \lambda = \frac{b}{a}$, and $\cos\phi = \lambda\cos\theta = \lambda\sin\phi = \frac{b}{a}\sin\phi$,

that is, $\quad \tan\phi = \frac{a}{b}$,

so that a common point of intersection is

$$\xi = \frac{ab}{(a^2+b^2)^{\frac{1}{2}}}, \quad \eta = \frac{ab}{(a^2+b^2)^{\frac{1}{2}}}.$$

Both the above solutions are included in the general formula

$$\frac{A_1B_1 - H_1^2}{(A_1+B_1)^2} = \frac{A_2B_2 - H_2^2}{(A_2+B_2)^2},$$

viz., we have $a^2 A_1 = 1, \quad b^2 B_1 = 1, \quad H_1 = 0,$

$$a^2\lambda^2 A_2 = 1, \quad b^2\lambda^{-2} B_2 = 1, \quad H_2 = 0,$$

whence $$\frac{1}{a^2\lambda^2} + \frac{\lambda^2}{b^2} = \frac{1}{a^2} + \frac{1}{b^2},$$

which is satisfied by $\lambda = 1$, $\lambda = \dfrac{b}{a}$.

To find the envelope of the system of ellipses

$$\frac{x^2}{a^2\lambda^2} + \frac{\lambda^2 y^2}{b^2} = 1,$$

we have $$a^2y^2 . \lambda^4 - a^2b^2 . \lambda^2 + b^2x^2 = 0,$$

the envelope of which is $\quad a^4b^4 = 4a^2b^2x^2y^2,$

which gives the pair of equilateral hyperbolas $xy = \pm \frac{1}{2}ab$.

Viewing the question in another light, we have to find the envelope of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{when} \quad ab = k^2 \quad \text{......................(3, 4).}$$

Differentiating (3) and (4), we get

$$\frac{x^2}{a^3}da + \frac{y^2}{b^3}db = 0, \quad b\,da + a\,db = 0,$$

whence, by the method of undetermined multipliers,

$$\frac{x^2}{a^3} = \mu b, \quad \frac{y^2}{b^3} = \mu a.$$

Substituting in (3), $$\mu = \frac{1}{2ab}.$$

Thus $$x^2 = \tfrac{1}{2}a^2, \quad y^2 = \tfrac{1}{2}b^2,$$

whence, attending to (4), $\quad xy = \pm \frac{1}{2}k^2,$

the same pair of equilateral hyperbolas as before.

3732. (By Professor BURNSIDE, M.A.)—Prove that the circle described on the axis major of a conic is its polar reciprocal.

Solution.

It can easily be proved from pure geometry or from pure analysis (BERTRAND's *Calcul.*, t. I., pp. 11, 81) that the reciprocal polar of any curve with regard to any given point is the inverse of the pedal of that curve with respect to that point; this, in fact, is only a different statement of the fundamental proposition in the theory of reciprocal polars, viz., if we take the inverse of the pedal of the inverse of the pedal of any curve, with respect to any one common origin, and any common constant of inversion, we reproduce the original curve itself. Hence, since the first focal pedal of a conic is the auxiliary circle, the polar reciprocal is the inverse of the auxiliary circle with regard to either focus; and, from the elementary theory of inversion, this inverse-curve is itself a circle, and, by properly selecting the constant of inversion, the inverse-circle may be made to coincide with the primitive, which proves the proposition that the

auxiliary circle is the polar reciprocal of the conic. This also follows analytically, viz., if we remove the origin to either focus, the auxiliary circle becomes $(x \pm ae)^2 + y^2 = a^2$, or $x^2 \pm 2aex + y^2 = b^2$. The equation of the inverse is got by substituting $\dfrac{k^2x}{x^2+y^2}$, $\dfrac{k^2y}{x^2+y^2}$, for x and y, viz., the inverse is $x^2 \mp \dfrac{2aek^2}{b^2}x + y^2 = \dfrac{k^4}{b^2}$; and, if we put $k^4 = b^4$, this coincides with the auxiliary circle.

Since the inverse of the circle $x^2+y^2+2gx+2fy+c=0$ is obviously the circle
$$x^2+y^2+\frac{2gk^2}{c}x+\frac{2fk^2}{c}y+\frac{k^4}{c}=0,$$
it is clear that the circle will invert into itself, if $k^2 = c$. Now, writing the equation in the form $(x+g)^2+(y+f)^2 = g^2+f^2-c$, the geometrical meaning of c is seen to be the square of the tangent drawn from the origin of inversion to the circle; hence we infer that a circle inverts into itself when, and only when, the radius of inversion is the tangent to the circle, and this is geometrically evident.

3734. (By J. J. Walker, M.A., F.R.S.)—If α, β, γ be any three quantities, prove that
$$\Sigma\alpha(\beta-\gamma)^2\,\Sigma(2\beta+2\gamma-\alpha)(\beta-\gamma)^2 - \Sigma\beta\gamma\,[\Sigma(\beta-\gamma)^2]^2$$
$$= 9(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2.$$

Solution.

Since α, β, γ are any three quantities, we may regard them as the roots of the arbitrary cubic equation $a_0x^3+3a_1x^2+3a_2x+a_3=0$. Then, from the ordinary theory of the symmetric functions of the roots, we have
$$a_0\Sigma\beta\gamma = 3a_2,\quad a_0^2\Sigma(\beta-\gamma)^2 = 18(a_1^2-a_0a_2) = -18\mathrm{H},$$
$$a_0^2\Sigma\alpha(\beta-\gamma)^2 = 9(a_0a_3-a_1a_2) = 9\mathrm{L},\ \text{say},$$
$$a_0^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = -27(\mathrm{G}^2+4\mathrm{H}^3).$$
Again,
$$\Sigma(2\beta+2\gamma-\alpha)(\beta-\gamma)^2 = \Sigma[2(\alpha+\beta+\gamma)-3\alpha](\beta-\gamma)^2$$
$$= 2(\alpha+\beta+\gamma)\Sigma(\beta-\gamma)^2 - 3\Sigma\alpha(\beta-\gamma)^2$$
$$= \frac{108a_1\mathrm{H}}{a_0^3} - \frac{27\mathrm{L}}{a_0^2} = \frac{27}{a_0^3}(2a_1\mathrm{H}-\mathrm{G}),$$
since obviously $a_0\mathrm{L} = 2a_1\mathrm{H}+\mathrm{G}$.

Therefore the sinister of the proposed identity
$$= \frac{9(2a_1\mathrm{H}+\mathrm{G})}{a_0^3}\cdot\frac{27(2a_1\mathrm{H}-\mathrm{G})}{a_0^3} - \frac{3a_2}{a_0}\cdot\frac{18^2.\mathrm{H}^2}{a_0^4}$$
$$= \frac{243}{a_0^6}(4a_1^2\mathrm{H}^2-\mathrm{G}^2-4a_0a_2\mathrm{H}^2)$$
$$= -\frac{243}{a_0^6}[\mathrm{G}^2+4\mathrm{H}^2(a_0a_2-a_1^2)] = -\frac{243}{a_0^6}(\mathrm{G}^2+4\mathrm{H}^3)$$
$$= 9(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2.$$

Of course, the truth of the identity can be shown by the well-known

method that, since for $\beta=\gamma$ the sinister vanishes, therefore $(\beta-\gamma)$ is a factor thereof [but why $(\beta-\gamma)^2$?] and similarly for the other factors; the numerical coefficient is then easily determined to be 9. But the above method, by symmetric functions, points out the way in which the identity was probably discovered.

3786 & 6806. (By D. WICKERSHAM, M.A.)—(3786.) Required the equation to the curve which cuts, at an angle of 45°, all ellipses having a common major axis.

(6806.) (By Professor ARTEMAS MARTIN, M.A., Ph.D.)—Required a complete solution of the differential equation,

$$\frac{dy}{dx}=\frac{a^2-xy-x^2}{a^2+xy-x^2}.$$

Solution.

Since every member of the family of ellipses has the same major axis, we may take the equation of any one of them to be

$$\phi\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \quad\text{.............................(1)};$$

and we shall first show how to find *any* trajectory of $\phi=0$. In fact, if α be the angle at which the system of trajectories cuts the family of ellipses, and $m=\cot\alpha$, we have to eliminate b between

$$\phi=0 \quad\text{and}\quad \frac{dy}{dx}=\left(\frac{d\phi}{dy}-m\frac{d\phi}{dx}\right)\Big/\left(m\frac{d\phi}{y}+\frac{d\phi}{dx}\right),$$

or, since $$\frac{d\phi}{dx}=\frac{2x}{a^2},\quad \frac{d\phi}{dy}=\frac{2y}{b^2},$$

the equations from which b has to be eliminated are

$$b^2=\frac{a^2y^2}{a^2-x^2},\quad \frac{dy}{dx}=\left(\frac{y}{b^2}-m\frac{x}{a^2}\right)\Big/\left(m\frac{y}{b^2}+\frac{x}{a^2}\right) \quad\text{............(2)}.$$

The result of the elimination is easily seen to be

$$\frac{dy}{dx}=\frac{a^2-x^2-mxy}{m(a^2-x^2)+xy} \quad\text{...............................(3)},$$

which is accordingly the differential equation to the system of trajectories. The integral equation can be obtained when the trajectories are orthogonal,

viz., we have $$\alpha=\tfrac{1}{2}\pi,\quad m=0,\ \text{and}\ \frac{dy}{dx}=\frac{a^2-x^2}{xy},$$

which gives $$x\,dx+y\,dy=\frac{a^2\,dx}{x},$$

whence, integrating, $x^2+y^2=2a^2\log kx$, where k is an arbitrary constant.

In the case when $\alpha=\tfrac{1}{4}\pi$, we have $m=1$, and (3) becomes

$$\frac{dy}{dx}=\frac{a^2-x^2-xy}{a^2-x^2+xy} \quad\text{.................................(4)},$$

and, as the variables cannot be separated, I do not see how this can be

integrated in a finite form. Besides, as all the tests, given by BOOLE, for the discovery of an integrating factor, fail in this case, I suspect that no finite integral exists. The integration, however, can be completely performed, as required in Quest. 6806, by means of a series; viz., assume

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \qquad \ldots\ldots\ldots\ldots(5),$$

so that

$$xy = a_0 x + a_1 x^2 + a_2 x^3 + \ldots + a_n x^{n+1} + \ldots,$$

$$\frac{dy}{dx} = a_1 + 2a_2 x + \ldots + na_n x^{n-1} + \ldots$$

Substituting in (4), we get

$$[a_1 + 2a_2 x + \ldots + na_n x^{n-1} + \ldots] \times [a^2 + a_0 x + (a_1 - 1)x^2 + \ldots + a_n x^{n+1} + \ldots]$$
$$= a^2 - a_0 x - (a_1 + 1)x^2 - a_2 x^3 - \ldots - a_n x^{n+1} - \ldots$$

Equating coefficients, we have

$$a^2 = a^2 a_1, \quad -a_0 = 2a^2 a_2 + a_0 a_1, \quad -(a_1 + 1) = 3a^2 a_3 + 2a_0 a_2 + a_1(a_1 - 1),$$
$$-a_2 = 4a^2 a_4 + 3a_0 a_3 + 2a_2(a_1 - 1) + a_1 a_2, \quad \&c. = \&c.,$$
$$-a_{n-1} = (n+1)a^2 a_{n+1} + na_0 a_n + (n-1)a_{n-1}(a_1 - 1)$$
$$+ (n-2)a_{n-2}a_2 + \ldots + 2a_2 a_{n-2} + a_1 a_{n-1}, \quad \&c. = \&c.$$

Hence we have the following values for the first five coefficients:—

$$a_1 = 1, \quad a_2 = -\frac{a_0}{a^2}, \quad a_3 = \tfrac{2}{3} \cdot \frac{a_0^2 - a^2}{a^4}, \quad a_4 = \frac{a_0(2a^2 - a_0^2)}{2a^6},$$
$$a_5 = \tfrac{2}{15}\frac{2a^4 - 17a_0^2 a^2 + 6a_0^4}{a^8}.$$

Substituting these values in (5), we obtain the required integral; it will be noticed that this final expression involves the arbitrary constant a_0, as it should, since (4) is a differential equation of the first order and degree. One particular case of interest may be noticed, viz., the equation of that one of the system of trajectories, which passes through the origin. Then

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = -\frac{2}{3a^2}, \quad a_4 = 0, \quad a_5 = \frac{4}{15a^4}, \ \&c.$$

Therefore the equation required is

$$y = x - \frac{2}{3a^2}x^3 + \frac{4}{15a^4} \cdot x^5, \quad \&c.$$

3811. (By A. ESCOTT, M.A., F.R.A.S.)—AB, AC are two given indefinite straight lines, touching a given conic in E and F; a variable line BC is drawn touching the conic, and BF, CE are drawn intersecting in P; show that the locus of P is another conic, intersecting the given lines AB, AC in E, F.

Solution.

Let the fixed tangents AB AC be L, M, and the equation of the conic referred to the two tangents and their chord is $LM = R^2$. Then, any tangent, as BC, may be written

$$\mu^2 L - 2\mu R + M = 0,$$

and the point of intersection of the line L with this tangent will have its

coordinates L, R, M respectively proportional to 0, 1, 2μ. Again, the equation of the line joining this point to *any* fixed point (L_1, R_1, M_1)—not necessarily the point of contact F—will be

$$LM_1 - L_1M = 2\mu\,(LR_1 - L_1R).$$

Similarly, the equation of the line joining the fixed point (L_2, R_2, M_2)—not necessarily the point of contact E—to the point $(2, \mu, 0)$, which is the point of intersection of the line M with the variable tangent, is

$$2\,(RM_2 - R_2M) = \mu\,(LM_2 - L_2M).$$

Eliminating μ, the locus of P is found to be

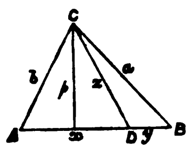
$$(LM_1 - L_1M)\,(LM_2 - L_2M) = 4\,(LR_1 - L_1R)\,(RM_2 - R_2M),$$

which represents a conic passing through the two given points. In the very particular case, when the given conic is a circle, and the given points the points of contact of the tangents to the circle, the locus-conic is easily seen to be an ellipse touching the given lines AB, AC in E, F.

3814. (By the Editor.)—If, in a triangle ABC, any line CD be drawn from C to AB, prove that, if p be the perpendicular from C on AB, and r, r_1, r_2 the in-radii of the triangles ABC, ADC, BDC, then

$$p\,(r_1 + r_2 - r) = 2r_1r_2 \quad \text{and} \quad (p - 2r_1)\,(p - 2r_2) = p\,(p - 2r).$$

Solution.



Let the lengths of the several lines be as indicated on the diagram; let r, Δ, s denote the in-radius, area, and semi-perimeter of the triangle ABC; r_1, Δ_1, s_1, r_2, Δ_2, s_2 the corresponding elements of the triangles ADC, BDC respectively. Then we have, evidently, the relations

$$\Delta = rs, \quad \Delta_1 = r_1s_1, \quad \Delta_2 = r_2s_2,$$

$$\left.\begin{aligned} 2\Delta &= p\,(x+y) \\ 2\Delta_1 &= px \\ 2\Delta_2 &= py \end{aligned}\right\}, \qquad \left.\begin{aligned} 2s &= a+b+x+y \\ 2s_1 &= b+x+z \\ 2s_2 &= a+y+z \end{aligned}\right\}.$$

Moreover, we have another relation from pure geometry, viz.,

$$a^2x + b^2y = x^2y + xy^2 + (x+y)\,z^2 = (x+y)\,(z^2 + xy).$$

(See Casey's *Sequel to Euclid*, p. 24, Prop. 9.)

By adding $(x+y)\,[ab + z\,(a+b)]$ to both sides of this identity, we obtain

$$\begin{aligned}[z^2 + z\,(a+b) + ab + xy]\,(x+y) &= a^2x + b^2y + ab\,(x+y) + z\,(a+b)(x+y) \\ &= [ax + by + z\,(x+y)]\,(a+b),\end{aligned}$$

whence
$$\frac{(z+a)\,(z+b) + xy}{ax + by + z\,(x+y)} = \frac{a+b}{x+y}.$$

Adding and subtracting numerators and denominators, we have

$$\frac{(b+z-x)(a+z-y)}{(b+z+x)(a+z+y)} = \frac{(a+b)-(x+y)}{(a+b)+(x+y)}.$$

To prove the first relation in question, we have

$$r = \frac{p(x+y)}{a+b+x+y}, \quad r_1 = \frac{px}{b+x+z}, \quad r_2 = \frac{py}{a+y+z},$$

which give $r_1 + r_2 - r = p\left\{\frac{x}{b+x+z} + \frac{y}{a+y+z} - \frac{x}{a+b+x+y} - \frac{y}{a+b+x+y}\right\}$

$$= p\left\{\frac{x(a+y-z)}{(b+x+z)(a+b+x+y)} + \frac{y(b+x-z)}{(a+y+z)(a+b+x+y)}\right\}$$

$$= \frac{p}{a+b+x+y}\left\{\frac{x\left[(a+y)^2-z^2\right]+y\left[(b+x)^2-z^2\right]}{(b+x+z)(a+y+z)}\right\}.$$

Now, by virtue of the geometrical theorem quoted above, the numerator of the fraction within brackets at once reduces to $2xy\,(a+b+x+y)$, which makes

$$r_1 + r_2 - r = \frac{2pxy}{(b+x+z)(a+y+z)}.$$

Therefore
$$p\,(r_1 + r_2 - r) = 2r_1 r_2,$$
which is exactly the first theorem in question.

To prove the second part, we have

$$p - 2r = p - \frac{2p\,(x+y)}{a+b+x+y} = p \cdot \frac{(a+b)-(x+y)}{(a+b)+(x+y)},$$

$$p - 2r_1 = p - \frac{2px}{b+x+z} = p \cdot \frac{b-x+z}{b+x+z},$$

$$p - 2r_2 = p - \frac{2py}{a+y+z} = p \cdot \frac{a-y+z}{a+y+z}.$$

Hence, attending to the identity proved above, we have

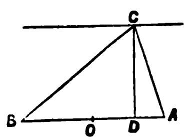
$$(p-2r_1)(p-2r_2) = p\,(p-2r),$$

which is the second theorem in question. Of course, this second relation is only another way of writing the first relation, as is evident on expanding it, when it reduces to the first relation; but I have thought it better to prove the relations by different and independent methods. The identity proved above may obviously be written in the form

$$\frac{s_1 s_2}{s} = \frac{(s_1-x)(s_2-y)}{s-(x+y)}.$$

3853 & 5396. (By the late Dr. Booth, F.R.S.)—Given the base and altitude of a triangle, find the envelope of the bisectors of the vertical angle.

Solution.



Let the given base BA ($=2a$) be taken as the axis of x, and its middle point O as origin. Then, as the altitude is given, the locus of the vertex is a right line parallel to the base, and at a distance b, equal to the given altitude, from it. Accordingly, the coordinates of A, B, C are $(a,\ 0)$, $(-a,\ 0)$, $(\xi,\ b)$ respectively, ξ alone being the variable

parameter which determines the required envelope of the bisector of the angle ACB.

Now, the equations of AC and BC are easily seen to be, respectively,

$$bx+(a-\xi)\,y-ab=0,\quad bx-(a+\xi)\,y+ab=0.$$

Therefore the equation of the bisectors of the angle between these lines, is at once written down to be

$$\frac{bx+(a-\xi)\,y-ab}{\left\{b^2+(a-\xi)^2\right\}^{\frac{1}{2}}}=\pm\frac{bx-(a+\xi)\,y+ab}{\left\{b^2+(a+\xi)^2\right\}^{\frac{1}{2}}}.$$

We have now to seek the envelope of this for different values of ξ. Squaring and arranging both sides of the equation, we get

$$\frac{\left\{(bx-\xi y)+a\,(y-b)\right\}^2}{\left\{(bx-\xi y)-a\,(y-b)\right\}^2}=\frac{(a^2+b^2+\xi^2)-2a\xi}{(a^2+b^2+\xi^2)+2a\xi},$$

which, by the substitutions

$$a^2+b^2+\xi^2=\mathrm{A},\quad bx-\xi y=\mathrm{P},\quad 2a\xi=\mathrm{B},\quad a\,(y-b)=\mathrm{Q},$$

assumes the form $\left(\dfrac{\mathrm{P}+\mathrm{Q}}{\mathrm{P}-\mathrm{Q}}\right)^2=\dfrac{\mathrm{A}-\mathrm{B}}{\mathrm{A}+\mathrm{B}}$, or $\dfrac{\mathrm{P}^2+\mathrm{Q}^2}{2\mathrm{PQ}}=-\dfrac{\mathrm{A}}{\mathrm{B}}$.

Restoring the values of A, B, P, Q, we get

$$\frac{(bx-\xi y)^2+a^2\,(y-b)^2}{2a\,(bx-\xi y)(y-b)}=-\frac{a^2+b^2+\xi^2}{2a\xi},$$

which, after some easy reductions, leads to the equation

$$y\xi^3-x\,(y+b)\,\xi^2+\left\{b\,(x^2-y^2)-(a^2-b^2)\,y+a^2b\right\}\xi+(a^2+b^2)\,x\,(y-b)=0.$$

This is of the form $\qquad a\xi^3+3\beta\xi^2+3\gamma\xi+\delta=0$ ....(1),

where $\alpha=y,\ \beta=-\frac{1}{3}x\,(y+b),\ \gamma=\frac{1}{3}\left\{b\,(x^2-y^2)-(a^2-b^2)\,y+a^2b\right\},$

$$\delta=(a^2+b^2)\,x\,(y-b).$$

Now, in order to find the envelope of (1), we differentiate it, and obtain

$$\alpha\xi^2+2\beta\xi+\gamma=0,\quad\text{or}\quad \alpha\xi^3+2\beta\xi^2+\gamma\xi=0\ldots\ldots\ldots\ldots(2,\,3).$$

Subtracting (3) from (1), we have

$$\beta\xi^2+2\gamma\xi+\delta=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Eliminating ξ between the quadratics (2) and (4), we have

$$4\,(\alpha\gamma-\beta^2)(\beta\delta-\gamma^2)=(\alpha\delta+\beta\gamma-2\beta\gamma)^2=(\alpha\delta-\beta\gamma)^2,$$

which, in its developed form, is

$$3\beta^2\gamma^2+6\alpha\beta\gamma\delta-\alpha^2\delta^2=4\,(\beta^3\delta+\gamma^3\alpha)\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

or

$$3\frac{\beta\gamma}{\alpha\delta}-\frac{\alpha\delta}{\beta\gamma}+6=4\left(\frac{\beta^2}{\alpha\gamma}+\frac{\gamma^2}{\beta\delta}\right)\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and this accordingly represents the envelope sought. Now, since α is of the first degree, and β, γ, δ of the second degree in x and y, it is clear from (5) that the envelope is a curve of the eighth degree; also, since for $x=0$, $y=0$, α, β, δ vanish identically, it is obvious that the envelope passes through the origin, which is the middle point of the given base. The actual equation in terms of x and y is found to be

$$\left\{b\,(x^2-y^2+a^2)-(a^2-b^2)\,y\right\}^2\left\{x^2\,(y+b)^2-4by\,(x^2-y^2+a^2)+4\,(a^2-b^2)\,y^2\right\}$$
$$+4\,(a^2+b^2)\,x^4\,(y-b)\,(y+b)^3$$
$$=27\,(a^2+b^2)^2\,x^2y^2\,(y-b)^2+18\,(a^2+b^2)\,x^2y\,(y^2-b^2)\left\{b\,(x^2-y^2+a^2)-(a^2-b^2)y\right\}.$$

This form is too hopelessly complicated to admit of any reduction. Even in the particular case when $a = b$, the degree of the equation is not reduced.

3899. (By H. MURPHY.)—Find the envelope of the circles described on the lines drawn from a focus to any point on a conic.

7288. (By R. F. SCOTT, M.A.)—Pairs of tangents are drawn to a closed plane oval curve (without singularities) inclined at a constant angle 2α. The internal and external angles between these tangents are bisected. If X, Y, P be the areas of the pedals, with respect to any point within the oval, of the envelopes of the internal and external bisectors and of the original curve, prove that $P = X\cos^2\alpha + Y\sin^2\alpha$.

8110. (By the EDITOR.)—If the pedal of any closed curve whose area is A, be taken with respect to an internal point, and through each point of the pedal a straight line be drawn making an angle α with the radius vector to that point; then B, the area of the curve enveloped by these lines, is given by the equation $B = A\sin^2\alpha$.

Solution, with Notes on the General Theory of Pedals.

1. If from any point as pole or origin, lines be drawn intersecting the successive tangents to a given curve, at a constant angle α, all in the *same* sense, the locus of the points of intersection of these lines and the tangents may be called a *pedal* of the curve with respect to the assumed origin. It is clear that all the pedals of the given curve are obtained, if we assign to α all possible values between 0 and π; so that, for facility of enumeration, any particular pedal may conveniently be specified as the α-pedal; since, when $2\alpha = \pi$, we get the right pedal, which is the ordinary pedal, this may be designated as the $\frac{1}{2}\pi$-pedal.

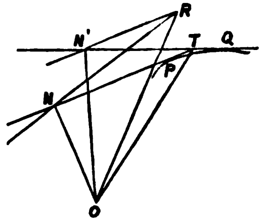
2. Let the equation to the $\frac{1}{2}\pi$-pedal be $\rho = f(\omega)$, where the radius vector ρ is the perpendicular from the origin on the tangent to the primitive curve, and ω is the angle made by ρ with the initial line, or the axis of x, which may be arbitrarily chosen. Then, if ρ_1, ω_1 be the corresponding elements of the oblique α-pedal, we have easily, from geometry, $\rho_1\sin\alpha = \rho$, $\omega_1 = \omega - \alpha + \frac{1}{2}\pi$, whence the equation of the α-pedal is seen to be $\rho\sin\alpha = f(\omega + \frac{1}{2}\pi - \alpha)$. By turning the initial line about its present position, through the complementary angle of α, this may be put into the form $\rho\sin\alpha = F(\omega)$.

If P be the area of the $\frac{1}{2}\pi$-pedal and Q that of the α-pedal, P and Q are easily connected. For, let $d\phi$ be the angle between two consecutive tangents to the primitive curve; then, from the theorem that the angle between any two lines is the same as the angle between any two other lines equally inclined to them in the *same* sense, we see that the angle between consecutive radii-vectores of the $\frac{1}{2}\pi$-pedal as well as of the α-pedal is $d\phi$. Hence

$$P = \tfrac{1}{2}\int\rho^2\,d\phi = \tfrac{1}{2}\sin^2\alpha\int\rho_1^{\,2}\,d\phi, \quad Q = \tfrac{1}{2}\int\rho_1^{\,2}\,d\phi,$$

which gives $P = Q\sin^2\alpha$.

3. We can now deduce geometrically the similar relation enunciated in question.



(8110.) Let O be the origin, ON, ON′ perpendiculars on two consecutive tangents. Draw NR and N′R, such that

$$\angle RNT = \angle RN'T = \tfrac{1}{2}\pi - \alpha,$$

whence R is a point on the envelope of NR. Then, as usual, ultimately when PQ is infinitely small, a circle passes through T, R, N′, N, and this circle obviously passes also through O, and has OT for a diameter. Let $OT = r$, $OR = r_1$. Then, from the theorem that any chord of a circle is equal to a diameter multiplied by the sine of the angle subtended at the circumference, we have, since $\angle ON'R = \pi - \alpha$, the relation $r_1 = r \sin\alpha$.

Therefore $\quad A = \tfrac{1}{2}\int r^2 d\theta, \quad B = \tfrac{1}{2}\int r_1^2 d\theta = \tfrac{1}{2}\sin^2\alpha \int r^2 d\theta,$

whence $B = A\sin^2\alpha$. If we make the α of this section equal to that of the last one, we get $A \times P = B \times Q$.

4. We shall now prove the theorem, that, if we take the envelope of the pedals of any curve, we get back the original curve itself. In fact, regarding the given curve as the envelope of its tangents, we may take the equation of the tangent in the form

$$x\cos\theta + y\sin\theta = f(\theta) \quad\text{.............................(1)},$$

where $f(\theta)$ signifies the perpendicular from the origin on the tangent, and the form of the function f depends solely on the nature of the given curve. Differentiating this equation, we have $-x\sin\theta + y\cos\theta = f'(\theta)$;

whence $\quad x^2 + y^2 = \rho^2 = f^2(\theta) + f'^2(\theta), \quad \tan(\omega - \theta) = \dfrac{f'(\theta)}{f(\theta)}$ ....(i.);

from which, by the elimination of θ, we get the equation of the given curve. Again, since the equation of the α-pedal has been shown to be

$$\rho\sin\alpha = f(\omega + \alpha - \tfrac{1}{2}\pi) \quad\text{..............................(2)},$$

the envelope of this is found by eliminating α from the system

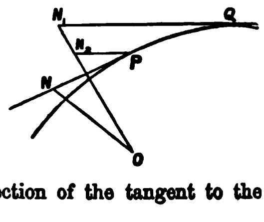
$$\rho\sin\alpha = f(\omega + \alpha - \tfrac{1}{2}\pi), \quad \rho\cos\alpha = f'(\omega + \alpha - \tfrac{1}{2}\pi),$$

whence

$$\rho^2 = f^2(\omega + \alpha - \tfrac{1}{2}\pi) + f'^2(\omega + \alpha - \tfrac{1}{2}\pi), \quad \tan\alpha = \frac{f(\omega + \alpha - \frac{1}{2}\pi)}{f'(\omega + \alpha - \frac{1}{2}\pi)} \quad\text{......(ii.)}.$$

Now, since by putting $\theta = \omega + \alpha - \tfrac{1}{2}\pi$ the systems (i.) and (ii.) coincide, their resultant must be the same, which shows that the given curve is the envelope of its pedals.

5. We shall next show that the problem of drawing a tangent to a pedal of any curve, is exactly the same problem as that of drawing a tangent to a circle; we shall hence deduce *immediately* a solution of question



(3899). As before, let O be the pedal-origin, and draw PN_2 parallel to QN_1, so that $\angle ON_2P = \angle ON_1Q = \angle ONP = \alpha$. Then, since ultimately N_1N_2 is an infinitesimal of the second order, we may use PN_2 instead of QN_1, so that NN_2 is the element-arc of the pedal; but it is clear that a circle passes through O, N, N_2, P, and ultimately N, N_2 coincide; so that the direction of the tangent to the

pedal at N coincides with the direction of the tangent to the circle at N. Thus the normal to the pedal at N is also the normal to the circle at that point. In the particular case of the $\frac{1}{2}\pi$-pedal, OP is obviously the diameter of the circle, and we have the following simple rule:—

"Draw any radius vector OP of the given curve, and draw ON perpendicular on the tangent; bisect OP in R, then NR is the normal to the pedal at N, and the tangent is the line through N at right angles to NR."

From the above geometrical considerations, it is manifest that the pedal of the curve is the envelope of the circles described on its radii-vectores. Hence the envelope required in Question 3899, is the first focal pedal of the conic, viz., the tangent being in the

central conic $y - mx = (a^2m^2 + b^2)^{\frac{1}{2}}$ | parabola $y = mx + am^{-1}$,

the line through the focus, at right angles to it, is

$$my + x = \pm\, ae \qquad \Big| \qquad y = -xm^{-1} + am^{-1};$$

whence, eliminating m, we get

$$x^2 + y^2 = a^2 \qquad \Big| \qquad x = 0,$$

which show that the envelope sought is, in one case, the auxiliary circle, in the other, the tangent at the vertex. If, however, the circle described on the focal radii-vectores are not semicircles, but contain an angle α, we have only to find the first focal oblique α-pedal, viz., as before, the tangents being $y - mx = (a^2m^2 + b^2)^{\frac{1}{2}}$ | $y = mx + am^{-1}$,

the lines through the foci, making an angle α with these, are found to be

$$m = \frac{gy + x \pm ae}{g\,(x \pm a) - y} \qquad \Big| \qquad m = \frac{gy + (x - a)}{g\,(x - a) - y},$$

where $g = \cot\alpha$. Eliminating m between the first pair, we find for the envelope, in the case of the central conic, the quartic curve

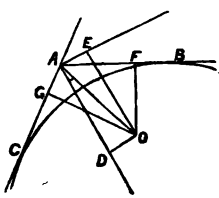
$$\left\{(x \pm ae)^2 + y^2\right\}\left\{x^2 + y^2 \mp 2aegy - (a^2 + b^2g^2)\right\} = 0,$$

which breaks up into a pair of infinitesimal circles (= the foci) and the pair of real circles $x^2 + (y \mp aeg)^2 = a^2\,(1 + g^2)$, the envelope of which, for different values of g, is the original conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. In the case of the parabola the envelope is the cubic curve

$$\left\{(x - a)^2 + y^2\right\}\left\{x + gy + ag^2\right\} = 0,$$

which, as in the preceding case, breaks up into an infinitesimal circle (= the focus) and the right line $x + gy + ag^2 = 0$, and the envelope of this pedal line for different values of g is $y^2 = 4ax$, the original parabola itself, just as was stated and proved in § 4. It is worth noticing that in each case the factor giving the infinitesimal focus-circle is extraneous, as it is independent of g.

6. We can now solve geometrically the pretty property enunciated in question



(7288). Let AB, AC be two tangents to the oval, including the constant angle 2α. Draw OF, OG perpendiculars on AB, AC, and OD, OE at right angles to the two bisectors. Then it is clear that OD, OE are the radii-vectores of the pedals of the envelopes of the bisectors. Join OA = δ. Let OD = p, OE = q, OF = r, OG = s, $\angle$OAD = θ. Then, if P_1, P_2 be the areas of the loci of F and G, it easily follows that, since obviously $d\phi$ (= angle of contingence of the

primitive curve) is the angle between any two consecutive radii-vectores of any of these pedals, we have

$$P_1 = \tfrac{1}{2}\int r^2\,d\phi,\quad P_2 = \tfrac{1}{2}\int s^2\,d\phi,\quad X = \tfrac{1}{2}\int p^2\,d\phi,\quad Y = \tfrac{1}{2}\int q^2\,d\phi,$$

where every integral is taken along the entire perimeter of the given non-singular oval. Now we know, from pure geometry, that $p = \delta\sin\theta$, $q = \delta\cos\theta$, $r = \delta\sin(\alpha-\theta)$, $s = \delta\sin(\alpha+\theta)$; therefore

$$P_1 = \tfrac{1}{2}\int \delta^2\sin^2(\alpha-\theta)\,d\phi,\quad P_2 = \tfrac{1}{2}\int \delta^2\sin^2(\alpha+\theta)\,d\phi;$$

whence

$$P_1 = \tfrac{1}{2}\int \delta^2\sin^2\alpha\cos^2\theta\,d\phi + \tfrac{1}{2}\int \delta^2\cos^2\alpha\sin^2\theta\,d\phi - \int \delta^2\sin\alpha\cos\alpha\sin\theta\cos\theta\,d\phi,$$

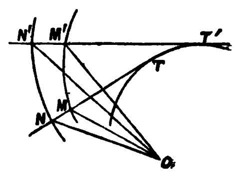
$$P_2 = \tfrac{1}{2}\int \delta^2\sin^2\alpha\cos^2\theta\,d\phi + \tfrac{1}{2}\int \delta^2\cos^2\alpha\sin^2\theta\,d\phi + \int \delta^2\sin\alpha\cos\alpha\sin\theta\cos\theta\,d\phi.$$

Adding, we get,

$$P_1 + P_2 = \sin^2\alpha\int \delta^2\cos^2\theta\,d\phi + \cos^2\alpha\int \delta^2\sin^2\theta\,d\phi = 2\,(X\cos^2\alpha + Y\sin^2\alpha).$$

Now, a moment's consideration shows that P_1 and P_2 are identically equal, since the integral represented by each of them is the whole area of the pedal of the given curve, as we may regard AB and AC as one and the same tangent in different positions. Hence $P_1 + P_2 = 2P$, and finally $P = X\cos^2\alpha + Y\sin^2\alpha$. Because $P_1 = P_2$, we deduce from the above equations that the integral $\int \delta^2\sin 2\theta\,d\phi$ ....(A), taken round the whole curve, vanishes. It is worth stating that this is equivalent to the fact that the integral $\int pq\,d\phi$, taken round the whole curve, vanishes.

7. We shall now evaluate, by the help of the above geometrical theory, two remarkable definite integrals, (B) and (C), similar to the one marked (A) above.



Let O be the pedal origin, and draw OM, OM′, making an angle α with the consecutive tangents TM, T′M′; also draw ON, ON′ making an angle $(\alpha + d\alpha)$ with the same tangents. Then it is obvious that all space outside the pedal may be divided into parallelogramic elements like MNN′M′. Let MNN′M′ $= d\sigma$, OM $= r$, TM $= t$. Then, since

$$d\sigma = \mathrm{NM}\,.\,\mathrm{MM}'.\sin \mathrm{NMM}',$$

and

$$\frac{\mathrm{NM}}{r} = \frac{d\alpha}{\sin\alpha},\quad \frac{\mathrm{MM}'}{t} = \frac{d\phi}{\sin \mathrm{NMM}'},$$

we have

$$d\sigma = \frac{rt}{\sin\alpha}\,d\phi\,d\alpha.$$

Therefore, we get

$$\int \frac{\sin\alpha}{t}\,\frac{d\sigma}{r} = \int_0^{2\pi}\int_0^{\pi} d\phi\,d\alpha = 2\pi^2 \quad ....\text{(B)}.$$

Again, we have

$$\int \frac{\sin^2\alpha\,.\,d\sigma}{t} = \int_0^{\pi}\int_0^{2\pi} r\sin\alpha\,d\alpha\,.\,d\phi = \int_0^{\pi}\int_0^{2\pi} d\alpha\,.\,f(\phi)\,d\phi,$$

where, as before, the equation of the tangent is

$$x\cos\phi + y\sin\phi = f(\phi),$$

and $f(\phi)$ is the perpendicular from the origin on the tangent. Hence

$$\int \frac{\sin^2 \alpha \,.\, d\sigma}{t} = \pi \int_0^{2\pi} f(\phi)\, d\phi = \pi \mathrm{L} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{C}),$$

by LEGENDRE's theorem, L being the total length of the curve. These two very general definite integrals may be compared with those evaluated by Professor CROFTON, by the aid of the Theory of Probabilities, in his celebrated memoir in the *Philosophical Transactions* for 1868.

3902. (By T. MITCHESON, B.A.)—If

$$\frac{\cos(\beta+\alpha)+\cos(\alpha+\gamma)}{\cos(\beta-\alpha)+\cos(\alpha-\gamma)} = \frac{\cos(\beta+\gamma)+\cos(\alpha+\gamma)}{\cos(\gamma-\beta)+\cos(\alpha-\gamma)},$$

prove that $\sin 2\beta = \mathrm{P} / \sin(\alpha-\gamma)$,

where $\mathrm{P} = \sin(\alpha+\gamma)\,[\sin(\beta-\gamma)+\sin(\alpha-\beta)+\sin(\alpha-\gamma)]$
$\quad -\sin(\alpha-\gamma)\,[\sin(\beta+\gamma)+\sin(\alpha+\beta)+\sin(\alpha+\gamma)]$.

Solution.

Adding and subtracting the numerators and denominators of the given identity, we have

$$\frac{\cos\alpha\,(\cos\beta+\cos\gamma)}{\sin\alpha\,(\sin\beta+\sin\gamma)} = \frac{\cos\gamma\,(\cos\beta+\cos\alpha)}{\sin\gamma\,(\sin\beta+\sin\alpha)},$$

which leads to

$$\frac{\cos\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\beta+\alpha)}{\sin\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta+\alpha)} = \frac{\sin\alpha\cos\gamma}{\cos\alpha\sin\gamma}.$$

Again, adding and subtracting numerators and denominators, we obtain

$$\frac{\sin(\alpha+\gamma)}{\sin(\alpha-\gamma)} = \frac{\sin\frac{1}{2}(\alpha+2\beta+\gamma)}{\sin\frac{1}{2}(\alpha-\gamma)}.$$

Then, if $\mathrm{Q} = \sin(\beta-\gamma)+\sin(\alpha-\beta)+\sin(\alpha-\gamma)$,

we have

$$\begin{aligned}\mathrm{Q} &= 2\sin\tfrac{1}{2}(\alpha-\gamma)\cos\tfrac{1}{2}(2\beta-\gamma-\alpha)+\sin(\alpha-\gamma)\\ &= 2\sin\tfrac{1}{2}(\alpha-\gamma)\,[\cos\tfrac{1}{2}(2\beta-\gamma-\alpha)+\cos\tfrac{1}{2}(\alpha-\gamma)].\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\sin(\alpha+\gamma)}{\sin(\alpha-\gamma)}\,\mathrm{Q}\\ &= \frac{\sin\frac{1}{2}(\alpha+2\beta+\gamma)}{\sin\frac{1}{2}(\alpha-\gamma)}\,.\,2\sin\tfrac{1}{2}(\alpha-\gamma)\,[\cos\tfrac{1}{2}(2\beta-\gamma-\alpha)+\cos\tfrac{1}{2}(\alpha-\gamma)]\\ &= 2\sin\tfrac{1}{2}(\alpha+2\beta+\gamma)\cos\tfrac{1}{2}(2\beta-\gamma-\alpha)+2\sin\tfrac{1}{2}(\alpha+2\beta+\gamma)\cos\tfrac{1}{2}(\alpha-\gamma)\\ &= \sin 2\beta+\sin(\alpha+\gamma)+\sin(\beta+\alpha)+\sin(\beta+\gamma).\end{aligned}$$

Hence $\sin(\alpha+\gamma)\,\mathrm{Q} = \sin(\alpha-\gamma)\sin 2\beta+\mathrm{R}\sin(\alpha-\gamma)$,

where $\mathrm{R} = \sin(\alpha+\gamma)+\sin(\beta+\alpha)+\sin(\beta+\gamma)$,

so that $\mathrm{Q}\sin(\alpha+\gamma)-\mathrm{R}\sin(\alpha-\gamma) = \mathrm{P}$,

and, finally, $\sin 2\beta = \mathrm{P} / \sin(\alpha-\gamma)$.

3955. (By the EDITOR.)—Find (1) the envelope of a series of circles, whose centres lie on any given curve, and which cut orthogonally the

circle of radius c around the origin as centre; and (2) of a circle passing through a given point, and whose centre moves along the circumference of a given circle.

Solution.

1. Consider any one of this family of circles; let a be its radius, δ the distance of its centre from the origin, and θ the angle which this centre-line makes with the initial line. Then, since this circle cuts orthogonally the circle $r = c$, we have $\delta^2 = a^2 + c^2$, and its equation must be of the form

$$\rho^2 - 2\delta\cos(\omega-\theta)\,\rho + \delta^2 - a^2 = 0,$$

where ρ, ω are the current polar coordinates. Moreover, as the centre is on the fixed curve, δ must satisfy the equation $\delta = f(\theta)$, which may be assumed to be the equation to the fixed curve. Hence the equation to the circle, whereof we have to find the envelope, becomes

$$\rho^2 - 2f(\theta)\cos(\omega-\theta)\,\rho + c^2 = 0.$$

Hence, differentiating, we have to determine θ in terms of ω from the equation
$$\tan(\theta-\omega) = \frac{f'(\theta)}{f(\theta)}.$$

Let $F(\omega)$ be the value of $f(\theta)\cos(\omega-\theta)$, when the value of θ in terms of ω is substituted in it from the above equation. Therefore the envelope of the series of circles is the pair of curves represented by the equation

$$\rho^2 - 2F(\omega)\,\rho + c^2 = 0.$$

2. In this particular case, the system of circles passes through the origin; so that, regarding the origin as a circle of infinitesimal radius, we have to put $c = 0$, whence the envelope reduces to the curve $\rho = 2F(\omega)$. To determine the form of the function F, we notice that, if the fixed curve on which the centres lie be itself a circle whose radius is k, and the distance of whose centre from the origin is b, we have the relation
$$\delta^2 - 2b\cos\theta\,.\,\delta + b^2 - k^2 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
in which, putting $\delta = f(\theta)$, and differentiating, we get

$$f(\theta)f'(\theta) - b\cos\theta\,.\,f'(\theta) + b\sin\theta\,.\,f(\theta) = 0.$$

On substituting $f'(\theta) = f(\theta)\tan(\theta-\omega)$, this becomes

$$f(\theta)\sin(\theta-\omega) = b\sin\omega,$$

whence $\delta = f(\theta) = \dfrac{b\sin\omega}{\sin(\theta-\omega)}$, and $F(\omega) = f(\theta)\cos(\omega-\theta) = \dfrac{b\sin\omega}{\tan(\omega-\theta)}$.

Substituting the above value of δ in (1), we get

$$b^2\sin^2\omega - 2b^2\sin\omega\cos\theta\sin(\theta-\omega) + (b^2-k^2)\sin^2(\theta-\omega) = 0,$$

which may be transformed into

$$[(b^2-k^2) + b^2\tan^2\omega]\tan^2\theta - 2(2b^2-k^2)\tan\omega\tan\theta + (4b^2-k^2)\tan^2\omega = 0,$$

from which equation $\tan\theta$ is known in terms of $\tan\omega$, and then $F(\omega)$ is known from the relation

$$\tan\theta = \frac{\tan\omega\,.\,F(\omega) - b\sin\omega}{F(\omega) + b\sin\omega\tan\omega} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Two particular cases of interest are

(i.) when $k^2 = 2b^2$, $\qquad \tan^2\theta = \dfrac{2\tan^2\omega}{1-\tan^2\omega}$,

(ii.) when $k^2 = 4b^2$, $\qquad \tan\theta = \dfrac{4\tan\omega}{3-\tan^2\omega}$.

The forms of the envelope in both these cases are exhibited easily. In the first case, put $F(\omega) = \frac{1}{2}\rho$ in (2), which gives

$$\tan\theta = \frac{\rho\tan\omega - 2b\sin\omega}{\rho + 2b\sin\omega\tan\omega} = \frac{\rho\sin\omega - 2b\sin\omega\cos\omega}{\rho\cos\omega + 2b\sin^2\omega},$$

whence $$\frac{2\tan^2\omega}{1-\tan^2\omega} = \left(\frac{\rho\sin\omega - 2b\sin\omega\cos\omega}{\rho\cos\omega + 2b\sin^2\omega}\right)^2,$$

which is accordingly the polar equation to the envelope, at once leading to the Cartesian form

$$(x^2-y^2)(x^2+y^2-2bx)^2 = 2\left[x(x^2+y^2)+2by^2\right]^2,$$

which reduces to the quartic

$$(x^2+y^2)^2 + 4bx(x^2+y^2) = 4b^2(x^2-2y^2);$$

and this is touched by the line bisecting the angle between the axes. In the second case, the relation

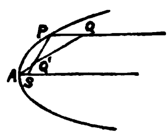
$$\tan\theta = \frac{4\tan\omega}{3-\tan^2\omega} \text{ gives } \tan(\omega-\theta) = -3\tan\omega,$$

whence $$\frac{b\sin\omega}{F(\omega)} = -3\tan\omega,$$

which leads to $F(\omega) = -3b\cos\omega$, so that $\rho = 2F(\omega) = -6b\cos\omega$. Therefore the polar equation to the envelope represents a circle whose Cartesian equation is $x^2+y^2+6bx = 0$.

3972. (By R. Tucker, M.A.)—On any diameter PQ of a parabola AP (vertex A, focus S), a distance PQ is taken. Show that Q′, the point of intersection of AQ, SP will trace out a circle, an ellipse, an hyperbola, or a parabola, according as PQ is taken equal to the abscissa, focal distance, or ordinate of the point P, or is constant.

Solution.



Let the parabola be $y^2 = 4ax$, so that AS $= a$, and take P to be (x_1, y_1), and Q to be (x_2, y_1), so that $y_1^2 = 4ax_1$; then the equations of PS and AQ are

$$(x_1 - a)y = y_1(x-a), \quad x_2 y = y_1 x,$$

so that, if (x, y) be the current coordinates of a point on the locus of Q′, we have

$$x = \frac{ax_2}{a+x_2-x_1} = \frac{a(\lambda+x_1)}{a+\lambda}, \quad y = \frac{ay_1}{a+x_2-x_1} = \frac{ay_1}{a+\lambda},$$

where $$\lambda = x_2 - x_1 = PQ.$$

We shall now consider the locus in the four particular cases.

(i.) $\lambda = x_1$. Therefore $$x = \frac{2ax_1}{a+x_1}, \quad y = \frac{ay_1}{a+x_1},$$

whence $$x_1 = \frac{ax}{2a-x}, \quad y_1 = \frac{2ay}{2a-x}.$$

Substituting in $y_1^2 = 4ax_1$, we get $x^2+y^2 = 2ax$, a circle whose centre is S and radius SA.

(ii.) $\lambda = PS = a + x_1$.

$\therefore\ x = \dfrac{a(a+2x_1)}{2a+x_1},\ y = \dfrac{ay_1}{2a+x_1}$, which give $x_1 = \dfrac{a(2x-a)}{2a-x},\ y_1 = \dfrac{3ay}{2a-x}$.

Substituting in $y_1^2 = 4ax_1$, we have $8x^2 - 20ax + 9y^2 + 8a^2 = 0$, an ellipse whose centre is on SA, at a distance $\frac{5}{4}a$ from A, and whose equation referred to the axes is $\frac{16}{9}x^2 + 2y^2 = a^2$.

(iii.) $\lambda = y_1$.

$\therefore\ x = \dfrac{a(x_1+y_1)}{a+y_1},\ y = \dfrac{ay_1}{a+y_1}$, which lead to $x_1 = \dfrac{a(x-y)}{a-y},\ y_1 = \dfrac{ay}{a-y}$.

Substituting in $y_1^2 = 4ax_1$, we get $3y^2 - 4xy - 4ay + 4ax = 0$, an hyperbola which passes through the vertex of the given parabola, and whose centre is the point $(\frac{1}{2}a, a)$.

(iv.) $\lambda = k$.

$\therefore\ x = \dfrac{a(k+x_1)}{a+k},\ y = \dfrac{ay_1}{a+k}$, whence $x_1 = \dfrac{(a+k)x - ak}{a},\ y_1 = \dfrac{(a+k)y}{a}$.

Substituting in $y_1^2 = 4ax_1$, we obtain $(a+k)^2y^2 = 4a^2(a+k)x - 4a^3k$, which represents a parabola.

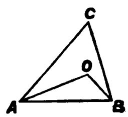
4016. (By T. Mitcheson, B.A.)—A triangle is inscribed in a circle upon a fixed base, and the vertex moves round the circumference of the circle. Find the locus of the in-centre.

Solution.

The question is obviously equivalent to this:—"Given the base and vertical angle of a triangle, find the locus of the in-centre."

Let O be the in-centre. Then

$\angle AOB = \pi - \frac{1}{2}\angle A - \frac{1}{2}\angle B = \pi - \frac{1}{2}(\pi - C) = \frac{1}{2}\pi + \frac{1}{2}C,$



which shows that the locus of O is a segment of circle on AB, on the same side as C, and that the angle subtending AB from the centre is the supplementary angle of C, the centre itself being on the circum-circle of ABC, or "that the segment on the other side of AB contains an angle $\frac{1}{2}(\pi + C)$.

4027. (By W. R. Nicoll, M.A.)—Two particles, projected with unequal initial velocities, have their horizontal ranges equal. Show that the tangents of their angles of projection are to one another in the duplicate ratio of their times of flight.

Solution.

Let u_1, θ_1, t_1, R_1 be the velocity, angle of projection, time of flight, and horizontal range, respectively, of the first projectile; denote the corresponding elements for the second projectile by changing the suffix

1 into 2. Then, we have

$$t_1 = \frac{2u_1 \sin\theta_1}{g},\ t_2 = \frac{2u_2 \sin\theta_2}{g},\ R_1 = \frac{2u_1^2 \sin\theta_1 \cos\theta_1}{g},\ R_2 = \frac{2u_2^2 \sin\theta_2 \cos\theta_2}{g}.$$

When $R_1 = R_2$, we must therefore have $\dfrac{u_1^2}{u_2^2} = \dfrac{\sin\theta_2 \cos\theta_2}{\sin\theta_1 \cos\theta_1}$. Hence

$$\frac{t_1^2}{t_2^2} = \frac{u_1^2}{u_2^2} \cdot \frac{\sin^2\theta_1}{\sin^2\theta_2} = \frac{\tan\theta_1}{\tan\theta_2},$$

which is exactly the theorem in question.

4028. (By J. F. MOULTON, F.R.S.)—A quantity of fluid fills a paraboloid of latus rectum c to a height h, the axis being vertical and vertex downwards. The density of the fluid varies as the depth. If the fluid pass into a vessel of the form generated round the axis of x, by the curve $a^4y^2 = 2ch^2x\,(a-x)(2a-x)$, where a is any constant, the density will vary as the square of the depth.

Solution.

Take the vertex of the paraboloid $y^2 = cx$ as origin. Let h_1 be the depth of the fluid in the new vessel. Then

$$\rho = \mu\,(h-x) = \mu_1\,(h_1-x_1)^2;$$

therefore $$x = h - \frac{1}{k}(h_1-x_1)^2, \text{ if } \mu = \mu_1 k.$$

Equating the volumes of corresponding strata, we have

$$cx\,dx = y_1^2 dx_1, \text{ which gives } k^2 y_1^2 = 2c\,(h_1-x_1)\,[hk-(h_1-x_1)^2].$$

Again, equating the total volumes, we have $h_1^2 = kh$. Therefore, finally, the generating curve of the new vessel is

$$h_1^4 y_1^2 = 2ch^2 x_1\,(h_1-x_1)\,(2h_1-x_1),$$

which agrees with the result in question, if we put $h_1 = a$. It ought to be noticed that the above method of proof not only shows that the curve in question satisfies the demands of the problem, but also that this is the *only* curve which possesses the property.

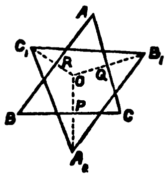
4088. (By W. SIVERLY.)—The sides of a triangle are reflective, and a luminous point is placed within it. Prove that the area of the triangle formed by the images varies as the rectangle contained by the segments of any chord of the circumscribing circle, passing through the luminous point.

Solution.

The property in question follows readily from the well-known theorem that, if through any point within a triangle parallels be drawn to the sides, the sum of the rectangles of their segments is equal to the rectangle contained by the segments of any chord of the circumcircle, passing through the given point. A purely geometrical proof of this theorem may

be seen in CASEY's *Elements of Euclid* (Ed. 1884, p. 231); and an analytical proof is implicitly given in SALMON's *Conics* (Ed. 1879, p. 89).

Take O to be the given point, and A_1, B_1, C_1 its images with respect to the sides. Let $OP = PA_1 = p$, $OQ = QB_1 = q$, $OR = RC_1 = r$; also let a, b, c be the sides of the given triangle, Δ its area, and Δ_1 the area of $A_1B_1C_1$. Then we have



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\Delta}{abc} = \lambda, \text{ say,}$$

$$\Delta_1 = 2(pq \sin C + qr \sin A + rp \sin B)$$

$$= 4\Delta\left(\frac{pq}{ab} + \frac{qr}{bc} + \frac{rp}{ca}\right).$$

Now, if through O we draw a line GOH parallel to BC, and take $OG = g$, $OH = h$, we have obviously $r = g \sin B$, $q = h \sin C$, so that

$$qr = gh \sin A \sin B = \lambda^2 . bc . gh, \quad \text{whence } \frac{qr}{bc} = \lambda^2 . gh.$$

Thus $$\Delta_1 = 4\Delta . \lambda^2 . M,$$

where M is the sum of the rectangles of the segments of the lines through O, drawn parallel to the sides of the triangle; hence, by the above theorem, the area of the image-triangle varies as the rectangle contained by the segments under any chord through O of the circumcircle of ABC.

4133. (By Professor WHITWORTH, M.A.)—The locus of a point, which moves so as to be always at a constant distance from its polar with respect to a given hyperbola, is a curve of the fourth order, having four real asymptotes, parallel, two and two, to the asymptotes of the hyperbola. The curve has double contact at infinity with the hyperbola, and cuts it again in four points, lying two and two, upon the polars of the circular points at infinity.

Solution.

Let the hyperbola be $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{......(1).}$$

Then, if (ξ, η) be the current coordinates of the point whose locus is sought, the polar of this, with respect to (1), is

$$\frac{\xi x}{a^2} - \frac{\eta y}{b^2} = 1 \quad \text{......(2).}$$

If the distance of (ξ, η) from (2) is always to be equal to a constant, say k, we must have $\left\{\frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} - 1\right\} \div \left\{\frac{\xi^2}{a^4} + \frac{\eta^2}{b^4}\right\}^{\frac{1}{2}} = k,$

whence $$\left(\frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} - 1\right)^2 = k^2\left(\frac{\xi^2}{a^4} + \frac{\eta^2}{b^4}\right) \quad \text{......(3),}$$

which shows that the required locus is a quartic.

In order to obtain the asymptotes of (3), we throw it into the form

$$\left(\frac{\xi^2}{a^2} - \frac{\eta^2}{b^2}\right)^2 - \left\{2\left(\frac{\xi^2}{a^2} - \frac{\eta^2}{b^2}\right) + k^2\left(\frac{\xi^2}{a^4} + \frac{\eta^2}{b^4}\right)\right\} + 1 = 0,$$

and notice that there is no term of the third degree in this equation, and the terms of the fourth degree form a perfect square. Hence, from the elementary proposition in the theory of plane curves, that when u_{n-1} vanishes identically, and the equation $u_n = 0$ leads to multiple roots, the *directions* of the asymptotes are the same as those of the lines $u_n = 0$, we see at once that the directions of the asymptotes of this quartic locus are given by
$$\left(\frac{\xi^2}{a^2} - \frac{\eta^2}{b^2}\right)^2 = 0.$$
From this it immediately follows that there are four real asymptotes, parallel, two and two, to the asymptotes of the hyperbola, which are given by
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Again, we may regard an asymptote as a tangent at infinity; hence, since two curves that touch a right line at the same point touch each other at that point, we see that the given hyperbola and the quartic touch each other at two points at infinity, and that
$$\frac{x}{a} + \frac{y}{b} = 0, \quad \frac{x}{a} - \frac{y}{b} = 0$$
represent the two common tangents to the two curves, at the two points of contact at infinity. This shows that the two curves have double contact at infinity. Since the curves must intersect in eight points, real or imaginary, and since four of these have been proved to lie at infinity, only four can be at a finite distance. It is then clear, from (1) and (3), that these four points are given as the roots of the quadratics
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^4} + \frac{y^2}{b^4} = 0,$$
which give for the actual values of the coordinates
$$x^2 = \frac{a^4}{a^2-b^2}, \quad y^2 = \frac{b^4}{a^2-b} \quad \text{.........................} (4).$$
Now, the polar of any point (α, β) with regard to the hyperbola is
$$\frac{\alpha x}{a^2} - \frac{\beta y}{b^2} = 1,$$
and, for the circular points at infinity, $\alpha = \pm\beta = \infty$, which gives, for the polars of these points, the pair of lines
$$\frac{x}{a^2} \pm \frac{y}{b^2} = 0 \quad \text{..................................} (5),$$
which obviously pass through the points given by (4).

4147. (By the Rev. A. F. Torry, M.A.)—P and Q are two points on an ellipse, so related that a ray of light, proceeding in direction PQ and being reflected at the curve at Q, will pass through the other extremity of the diameter through P. Prove that, if θ, ϕ be the eccentric angles at P, Q, we have $\tan\theta\tan\phi = -\dfrac{b^2}{a^2}$.

Solution.

Let θ, ϕ be the eccentric angles at P and Q respectively, and R the other extremity of the diameter through P. Then the coordinates of P, Q, R are $(a\cos\theta,\ b\sin\theta)$, $(a\cos\phi,\ b\sin\phi)$, $(-a\cos\theta,\ -b\sin\theta)$. Hence, if the equations of PQ, QR, and of the tangent at Q, be respectively

$$y = m_1 x + b_1,\quad y = m_2 x + b_2,\quad y = mx + b,$$

we easily obtain $m_1 = \dfrac{b}{a}\cdot\dfrac{\sin\theta-\sin\phi}{\cos\theta-\cos\phi} = -\dfrac{b}{a}\cot\dfrac{\theta+\phi}{2}$,

$$m_2 = \frac{b}{a}\,\frac{\sin\theta+\sin\phi}{\cos\theta+\cos\phi} = \frac{b}{a}\tan\frac{\theta+\phi}{2},\quad m = -\frac{b}{a}\,\frac{\cos\phi}{\sin\phi} = -\frac{b}{a}\cot\phi.$$

From these values, we at once deduce that

$$m_1 + m_2 = \frac{b}{a}\left(\tan\frac{\theta+\phi}{2} - \cot\frac{\theta+\phi}{2}\right) = -2\frac{b}{a}\,.\cot(\theta+\phi),$$

$$1 - m_1 m_2 = 1 + \frac{b^2}{a^2}.$$

Now, since the ray is reflected at the curve at Q, the incident and reflected rays are equally inclined to the normal at Q, or, in other words, the tangent may be regarded as the external bisector of the angle PQR; again, since it is clear that the angle made by the bisector with the axis of y is half the algebraic sum of the angles made with this axis by the lines PQ, QR themselves, we obtain, by equating tangent of twice the angle to tangent of sum, the relation $\dfrac{2m}{1-m^2} = \dfrac{m_1+m_2}{1-m_1m_2}$. Substituting the values of m, m_1, m_2 as given above, we have, after a slight reduction

$$\left(1 - \frac{b^2}{a^2}\cot^2\phi\right)\cot(\theta+\phi) = \left(1 + \frac{b^2}{a^2}\right)\cot\phi,$$

whence $$\frac{b^2}{a^2}\cot\phi = \frac{\cot(\theta+\phi)-\cot\phi}{1+\cot(\theta+\phi)\cot\phi} = -\tan\theta.$$

Therefore, finally, $$\tan\theta\tan\phi = -\frac{b^2}{a^2}.$$

4327. (By the Rev. W. Roberts, M.A.)—Find the locus of points on the ellipses of a confocal system, at which the perpendicular drawn from the centre on the tangent is in a constant ratio to the perpendicular from the centre on the line joining the extremities of the axes. Also find the system of curves cutting orthogonally the system which is obtained by supposing the above-mentioned constant ratio to assume different values.

Solution.

Let the equation to one of the confocals be $\dfrac{x^2}{A^2}+\dfrac{y^2}{B^2}=1$ ....(1),

in which, for the moment, I have written $a^2+\lambda^2 = A^2$, $b^2+\lambda^2 = B^2$, where a, b are the semi-axes of the primitive ellipse, and λ the parameter

of the confocal system. Then, if ϕ be any point (ξ, η) on (1), we have $\xi = A\cos\phi$, $\eta = B\sin\phi$, and the tangent is

$$\frac{x}{A}\cos\phi + \frac{y}{B}\sin\phi = 1 \quad\text{.........................(2),}$$

so that the perpendicular from the centre on (2) is

$$p_1 = \left(\frac{\cos^2\phi}{A^2} + \frac{\sin^2\phi}{B^2}\right)^{-\frac{1}{2}}.$$

As the equation of the line joining the extremities of the axes is

$$\frac{x}{A} + \frac{y}{B} = 1 \quad\text{..................................(3),}$$

the perpendicular from the centre on (3) is

$$p_2 = \left(\frac{1}{A^2} + \frac{1}{B^2}\right)^{-\frac{1}{2}}.$$

Hence, if $p_1 = kp_2$, where k is the given constant ratio, we have

$$k^2 = \frac{A^2 + B^2}{A^2\sin^2\phi + B^2\cos^2\phi}.$$

Substituting for A, B in terms of a, b, λ, we easily deduce the equations

$$\xi^2 = (a^2+\lambda^2)\cos^2\phi, \quad \eta^2 = (b^2+\lambda^2)\sin^2\phi \quad\text{............(4, 5),}$$

$$k^2(a^2\sin^2\phi + b^2\cos^2\phi + \lambda^2) = a^2 + b^2 + 2\lambda^2 \quad\text{..................(6),}$$

whence $\qquad (k^2-2)\lambda^2 = a^2(1-k^2\sin^2\phi) + b^2(1-k^2\cos^2\phi).$

Substituting this value of λ^2 in (4) and (5), we get

$$\frac{k^2-2}{a^2-b^2}\xi^2 = k^2\cos^4\phi - \cos^2\phi, \quad \frac{k^2-2}{a^2-b^2}\eta^2 = -k^2\sin^4\phi + \sin^2\phi \quad\text{......(7, 8).}$$

Therefore, putting $a^2-b^2 = c^2$, we have, from (7) and (8),

$$\frac{k^2-2}{k^2-1}(\xi^2+\eta^2) = c^2\cos 2\phi,$$

and $\qquad (k^2-1)c^2 - (k^2-2)(\xi^2-\eta^2) = \frac{1}{2}k^2c^2\sin^2 2\phi;$

whence ϕ is at once eliminated, and the resultant is

$$\left(\frac{k}{k^2-1}\right)^2(k^2-2)(\xi^2+\eta^2)^2 - 2c^2(\xi^2-\eta^2) + c^4 = 0 \quad\text{...........(9),}$$

which is the required equation to the locus sought. The locus is, accordingly, a Cassinian oval, the locus of the vertex of a triangle whose base and the rectangle under the sides are, respectively,

$$\frac{2c}{(k^2-2)^{\frac{1}{2}}}\left(k - \frac{1}{k}\right), \quad \frac{2c}{k(k^2-2)^{\frac{1}{2}}}.$$

There is a reduction in the degree of the locus, only when $k^2 = 2$, and in that case the locus degenerates into the equilateral hyperbola $\xi^2 - \eta^2 = \frac{1}{2}c^2$. It is worthy of remark that the locus does not in any way depend on the magnitude of the axes of the primitive ellipse, but simply on the distance between and the position of the foci.

We next proceed to determine the system of curves which cuts orthogonally the system obtained by varying k in (9). Now, in order to find the orthogonal trajectory of $\phi(x, y, f) = 0$, where f is the variable para-

meter, we have to eliminate f between

$$\phi = 0, \quad \frac{d\phi}{dy}dx = \frac{d\phi}{dx}dy.$$

As in the present equation k is involved only in the coefficient of the first term, we may write the equation in the form

$$\phi = f(\xi^2+\eta^2)^2 - 2c^2(\xi^2-\eta^2) + c^4 = 0 \text{..................} (10).$$

Hence we have to eliminate f between

$$\phi = 0, \quad \frac{d\phi}{d\xi}d\eta = \frac{d\phi}{d\eta}d\xi,$$

that is, between (10) and $f(\xi^2+\eta^2)(\xi d\eta - \eta d\xi) = c^2(\xi d\eta + \eta d\xi)$. The result of the elimination is

$$\eta(3\xi^2-\eta^2-c^2)\,d\xi + \xi(-\xi^2+3\eta^2+c^2)\,d\eta = 0 \text{ ....} (11),$$

which is, accordingly, the differential equation to the orthogonal system. To integrate (11), put it into the form $Md\xi + Nd\eta = 0$, and observe that

$$M = 3\xi^2\eta - \eta^3 - c^2\eta, \quad N = -\xi^3 + 3\xi\eta^2 + c^2\xi$$

lead to $\quad \xi^2\left(\frac{dN}{d\xi} - \frac{dM}{d\eta}\right) - 2N\xi = -4\xi^4, \quad M\xi + N\eta = 2\xi\eta(\xi^2+\eta^2),$

which give $\quad F(v) = \dfrac{-4\xi^4}{2\xi\eta(\xi^2+\eta^2)} = -\dfrac{2}{v(1+v^2)}, \quad$ where $\eta = v\xi$.

Hence we see, as in Boole, p. 77, that there exists an integrating factor μ, of the degree (-2), viz.,

$$\mu = \xi^{-2}\exp.\int F(v)\,dv = \frac{\xi^2+\eta^2}{\xi^2\eta^2}.$$

Multiplying the differential equation by this factor, and integrating, we find as the primitive equation

$$\left(\frac{\xi^3}{\eta} + \frac{\eta^3}{\xi}\right) - c^2\left(\frac{\xi}{\eta} - \frac{\eta}{\xi}\right) + 2\xi\eta + D = 0,$$

which involves the arbitrary constant D, and may be thrown into the form $(\xi^2+\eta^2)^2 - c^2(\xi^2-\eta^2) + D\xi\eta = 0$, which is accordingly the equation to the system of orthogonal trajectories of the Cassinoidal loci. One of the trajectories which all pass through the origin, is a Lemniscate of Bernoulli; for, when $D = 0$, the equation reduces to $(\xi^2+\eta^2)^2 = c^2(\xi^2-\eta^2)$, which in polar coordinates becomes $r^2 = c^2\cos 2\theta$. As has been remarked by Boole (p. 84), we can integrate (11) by writing it in the form

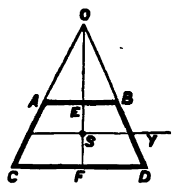
$$P_1 d\xi + P_2 d\eta + Q(xdy - ydx) = 0.$$

But, in the present instance, the discovery of the integrating factor leads to the easier method of the two.

*** *The following solutions are added as supplementary to what has been given in the Appendix to Volume XLIII.*

7586. (By Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—A trapezoid has two of its sides (l, m) parallel, and the other two equal; if the distance between the parallel sides be k, prove that (1) the equation of the maximum inscribed ellipse is $\frac{x^2}{k^2} + \frac{y^2}{lm} = \frac{1}{4}$; and show (2) how to construct the ellipse.

Solution by J. A. Owen, B.Sc.



1. Let ABCD be the trapezoid, AB $= l$, CD $= m$; let CA and DB be produced to meet at O; AC = BD; from O draw OEF perpendicular to AB; EF $= k$.

Take OC and OD for axes of x and y; the equation to a conic touching the axes is

$$(\alpha x + \beta y - 1)^2 = \lambda xy \text{ ....(1),}$$

where $\alpha x + \beta y - 1 = 0$ is the chord of contact. Let equations to AB and CD be $px + py = 1$ and $qx + qy = 1$. Then

$$(\alpha x + \beta y - px - py)^2 = \lambda xy \text{ ....(2)}$$

represents the two straight lines through to the points of intersection of the conic (1) with AB; therefore, since AB is to touch the conic, the lines must coincide; therefore

$$\{2(\alpha - p)(\beta - p) - \lambda\}^2 = 4(\alpha - p)^2(\beta - p)^2,$$

therefore $$\lambda = 4(\alpha - p)(\beta - p) \text{ ....(3).}$$

Similarly $\lambda = 4(\alpha - q)(\beta - q)$ for the line CD,

therefore $$p + q = \alpha + \beta \text{..................................(4).}$$

Equation (1) expanded is

$$\alpha^2 x^2 + xy(2\alpha\beta - \lambda) + \beta^2 y^2 - 2\alpha x - 2\beta y + 1 = 0,$$

and when equation to the ellipse is written in the usual form

$$(a, b, c, f, g, h \between x, y, 1),$$

the area is given by $\frac{\pi c \sin\omega}{(ab - h^2)^{\frac{3}{2}}}$. Making the alterations, we have

$$\text{area} = \frac{\pi \sin\omega}{\{\alpha^2\beta^2 - [\frac{1}{2}(2\alpha\beta - \lambda)]^2\}^{\frac{3}{2}}};$$

therefore $$u = \left(\frac{\pi \sin\omega}{\text{area}}\right)^2 = \{\alpha^2\beta^2 - [\tfrac{1}{4}(2\alpha\beta - \lambda)^2]\}^3$$

$$= [\tfrac{1}{4}(4\alpha\beta\lambda - \lambda^2)]^3 = \frac{\lambda^3}{64}(4\alpha\beta - \lambda)^3;$$

$$\therefore \quad du = \tfrac{1}{64}[3\lambda^2(4\alpha\beta - \lambda)^3 d\lambda + 3\lambda^3(4\alpha\beta - \lambda)^2(4\alpha d\beta + 4\beta d\alpha - d\lambda)] = 0;$$

therefore $$(4\alpha\beta - \lambda)\,d\lambda + \lambda(4\alpha d\beta + 4\beta d\alpha - d\lambda) = 0,$$

$$(2\alpha\beta - \lambda)\,d\lambda + \lambda(2\alpha d\beta + 2\beta d\alpha) = 0 \text{ ....(5).}$$

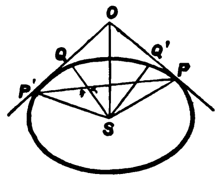
From (4) we have $d\alpha + d\beta = 0$. From (3) we have

$$d\lambda = 4\left[d\alpha(\beta - p) + d\beta(\alpha - p)\right] = 4(\beta d\alpha + \alpha d\beta) = 4(\beta - \alpha)\,d\alpha.$$

Substitute in (5), therefore

$$4(2\alpha\beta - \lambda)(\beta - \alpha)\,d\alpha + 2\lambda(\beta - \alpha)\,d\alpha = 0;\ \text{therefore}\ (\beta - \alpha)(4\alpha\beta - \lambda) = 0.$$

$4\alpha\beta = \lambda$ leads to no solution, as may be seen by substituting in (3); therefore we have $\alpha = \beta$ for a maximum. Therefore the line $\alpha x + \beta y = 1$ is equally inclined to OC and OD, is therefore parallel to AB and CD, and is bisected by OF at right angles.



Because OP = OP′; therefore SQ = SQ′, diameters parallel to tangents. But $SP^2 + SQ^2 = SP' + SQ'^2$ (conjugate diameters), therefore SP = SP′.

But equal diameters are equally inclined to axes; and angle OSP = OSP′, therefore OS must coincide with one of the axes; therefore centre of ellipse S must be mid-point of EF, and the ellipse must touch AB and CD at E and F.

To find the equation in the required form, take SO for axis of x and SY for axis of y. Let equation be

$$\frac{x^2}{(\frac{1}{2}k)^2} + \frac{y^2}{(\frac{1}{2}h)^2} = 1, \quad i.e., \quad \frac{x^2}{k^2} + \frac{y^2}{h^2} = \frac{1}{4},$$

where k is EF, and h is to be found.

The equation to BD is $\quad y = -\frac{(\frac{1}{2}m - \frac{1}{2}l)}{k}\,.\,x + \frac{1}{4}(l+m),$

that is, $$2(m-l)\,x + 4ky - k(l+m) = 0.$$

The condition that $\lambda x + \mu y - p = 0$ should touch $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ is

$$\lambda^2\alpha^2 + \mu^2\beta^2 = p^2;$$

therefore we have $\quad 4(m-l)^2\,.\,\frac{1}{4}(k^2) + 16k^2\,.\,\frac{1}{4}(k^2) = k^2(l+m)^2;$

$$\therefore\ (m-l)^2 + 4h^2 = (l+m)^2, \quad \therefore\ 4h^2 = 4lm, \quad \therefore\ h^2 = lm;$$

therefore $$\frac{x^2}{k^2} + \frac{y^2}{lm} = \frac{1}{4}.$$

To construct the ellipse; if $k^2 > lm$, then SE is the major axis; then mark off on EF from SE, parts equal to the difference of the squares of SE and the mean proportional of EB and F; then these points will be the foci, &c. If $k^2 < lm$, do similarly on SY.

8124. (By Professor Cochez.)—Trouver une courbe dont le rapport de son rayon de courbure à sa normale soit égal à 1 : μ.

Solution by Arthur Hill Curtis, LL.D., D.Sc.

Taking axes rectangular, and for X the line on which the normal N is intercepted, denoting by γ the radius of curvature, and by θ the angle at which a tangent to the curve is inclined to X, the condition of the

question gives the equation $\pm \frac{y}{\cos\theta} = \pm N = \mu\gamma = \mu\frac{ds}{d\theta}$; the upper sign being taken in case (1), where the radius of curvature and normal are in *opposite* directions, and the lower sign in case (2), where they are *co-directional*, or as $dy = \frac{ds}{\sin\theta}$, $\frac{\mu\, dy}{y} = \pm\frac{\sin\theta\, d\theta}{\cos\theta} = \mp\frac{d\cos\theta}{\cos\theta}$;

the integrals of which are

$$y^\mu \cos\theta = k^\mu \text{ in case (1)}, \quad \frac{y^\mu}{\cos\theta} = k^\mu \text{ in case (2)};$$

but $\frac{dy}{dx} = \tan\theta$; therefore, in (1), $\frac{k^\mu\, dy}{(y^{2\mu}-k^{2\mu})^{\frac{1}{2}}} = dx$.

The integration cannot be completed until the value of μ has been assigned; *e.g.*, if $\mu = 1$,

$$(a) \quad \frac{dy}{(y^2-k^2)^{\frac{1}{2}}} = \frac{dx}{k}, \quad \text{therefore} \quad y + (y^2-k^2)^{\frac{1}{2}} = ke^{\frac{x+A}{k}},$$

from which $$y-(y^2-k^2)^{\frac{1}{2}} = ke^{-\frac{x+A}{k}};$$

therefore, by addition, $y = \frac{1}{2}k\left(e^{\frac{x+A}{k}} + e^{-\frac{x+A}{k}}\right)$, the catenary;

(*b*) if $\mu = \frac{1}{2}$, $\frac{dy}{(y-k)^{\frac{1}{2}}} = \frac{dx}{k^{\frac{1}{2}}}$, therefore $(y-k)^{\frac{1}{2}} = \frac{x+a}{2k^{\frac{1}{2}}}$;

and therefore $4k(y-k) = (x+a)^2$, a parabola, the axis of X being the directrix.

In case (2) $\frac{dy}{dx} = \tan\theta = \frac{(k^{2\mu}-y^{2\mu})^{\frac{1}{2}}}{y^\mu}$, or $dx = \frac{y^\mu\, dy}{(k^{2\mu}-y^{2\mu})^{\frac{1}{2}}}$.

(a') If $\mu = 1$, $dx = \frac{y\, dy}{(k^2-y^2)^{\frac{1}{2}}}$, therefore $-(x-A) = (k^2-y^2)^{\frac{1}{2}}$,

therefore $$(x-A)^2 + y = k^2, \text{ a circle.}$$

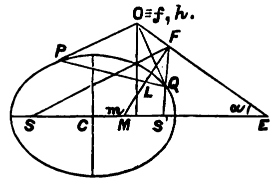
(b') If $\mu = \frac{1}{2}$, $$dx = \frac{y^{\frac{1}{2}}\, dy}{(k-y)^{\frac{1}{2}}} = \frac{y\, dy}{(ky-y^2)^{\frac{1}{2}}} = \frac{y\, dy}{\left[(\frac{1}{2}k)^2-(\frac{1}{2}k-y)^2\right]^{\frac{1}{2}}} = \frac{\left[\frac{1}{2}k-(\frac{1}{2}k-y)\right]dy}{\left[(\frac{1}{2}k)^2-(\frac{1}{2}k-y)^2\right]^{\frac{1}{2}}},$$

and therefore $(A-x) = \frac{1}{2}k\cos^{-1}\left(\frac{\frac{1}{2}k-y}{\frac{1}{2}k}\right) + \left[(\frac{1}{2}k)^2-(\frac{1}{2}k-y)^2\right]^{\frac{1}{2}}$, a cycloid generated by a circle of radius $\frac{1}{2}k$ rolling on X. [See Appendix to Vol. XLIII., p. 147.]

8129. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Given a point O and a system of confocal conics (foci S, S′, centre C), if OP, OQ be tangents to any one of these conics, and through each point of PQ there be drawn a straight line perpendicular to its polar with respect to this conic; prove that (1) the envelope of all such straight lines is definite (the parabola which is also the envelope of PQ and of the normals at P and Q);

(2) the locus of the point where each straight line meets its polar is also definite (being the circular cubic which the locus of P, Q and of the foot of the perpendicular from O on PQ); (3) this locus and envelope depend only upon the relative positions of O, S, S′, although there are in each case two parameters involved, which we may take to be a/b, the ratio of the axes of the conic, and Y'/X', where $(X'Y')$ is the point on PQ through which the perpendicular is drawn.

Solution by Arthur Hill Curtis, LL.D., D.Sc.



As the locus of the poles of any given line FE, denoted by T, taken with regard to a system of confocal conics, is a straight line perpendicular to T, it follows that this locus will be the perpendicular let fall on T from L, the pole of T with regard to *any one* conic of the system; thereby showing, *à priori*, that the envelope of the perpendicular let fall on *any system of lines* from their corresponding poles will depend, so far as the conics are concerned, *only* on a^2-b^2. F, the intersection of T with the perpendicular let fall on it from L, will be the point of contact of T with the particular conic of the system to which T is a tangent, and to which therefore FL is a normal; if, therefore, F be joined to the two foci S, S′, FL and FE will bisect the internal and external angles of the triangle SFS′, therefore SE is cut harmonically in M and S′. If then we consider a system of lines all passing through a given point O, whose coordinates with regard to the axes of the confocal system are f and h, we shall have the condition $EC \cdot CM = CS'^2$, or $(f+h\cot\alpha)\,m = c^2$, where $\alpha = \angle FES$, $m = CM$, and $c = CS = CS'$; the equation of FM therefore is

$$x\cos\alpha - y\sin\alpha - m\cos\alpha = 0, \text{ or } x\cos\alpha - y\sin\alpha - \left(\frac{c^2}{f+h\cot\alpha}\right)\cos\alpha = 0,$$

or
$$x\cot\alpha - y - \frac{c^2}{f+h\cot\alpha}\cot\alpha = 0,$$

or
$$hx\cot^2\alpha - (hy - fx + c^2)\cot\alpha - fy = 0,$$

the envelope of which is $\quad 4fhxy + (hy - fx + c^2)^2 = 0,$

or
$$(hy+fx)^2 + 2c^2(hy-fx) + c^4 = 0,$$

a parabola, which depends only on the three quantities f, h, and c, and the position of the centre and axes of the confocal system. As it has been shown that the envelope in question is touched by the corresponding normal to every line through the point O, at its point of contact with the particular conic of the system which can be described to touch this line, it is plainly touched by the normals at P and Q to the particular conic referred to in the question. Again, as the normal to a conic at a point of intersection with a confocal one is a tangent to the latter, the curve under consideration is also the envelope of the tangents to a system of confocal conics, the corresponding normals to which pass through the given point O. As PQ is a polar of O, the perpendicular from O on PQ is such a normal, and PQ such a tangent, and therefore touches the envelope.

The equation of the curve in question referred to the normals to the two confocals of the system through O, which bisect the angles between OC, OC′, may be easily found thus: it may be regarded (*Reprint*,

Vol. XLII., p. 83) as the envelope of the perpendiculars erected at the extremities of the radii vectores of the curve $r = \dfrac{k \sin\theta \cos\theta}{\sin(\theta+\alpha)}$, the equation of which system of lines is $x\cos\theta + y\sin\theta - \dfrac{k\sin\theta\cos\theta}{\sin(\theta+\alpha)} = 0$, which is equivalent to

$$(x\cos\alpha + y\sin\alpha - k)\sin 2\theta + (x\sin\alpha - y\cos\alpha)\cos 2\theta + (x\sin\alpha + y\cos\alpha) = 0,$$

the envelope of which is

$$(x\cos\alpha + y\sin\alpha - k)^2 + (x\sin\alpha - y\cos\alpha)^2 = (x\sin\alpha + y\cos\alpha)^2,$$

or $$(x\cos\alpha - y\sin\alpha)^2 - 2k(x\cos\alpha + y\sin\alpha) + k^2 = 0.$$

[See Appendix to Vol. XLIII., p. 148.]

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CORRIGENDA.

VOL. XXXVIII.

Page 12, line 4, *for* y *read* y^2.
,, 19, last line, *for* b *read* b^2.
,, 47, line 3, *for* equation *read* stable equilibrium.
,, 57, line 2, *for* b *read* b^6.
,, 65, line 20, *for* difficulty *read* differential.
,, 75, line 14, *for* $\sin n$ *read* $\sin n\theta$.
,, 75, line 15, *for* $r^2 - ar\cos\theta - a^2$ *read* $r^2 - 2ar\cos\theta + a^2$.
,, 120, last line, *for* 2_n *read* 2^n.

VOL. XXXIX.

Page 8, line 20, *for* 6648 *read* 6628.
,, 16, line 8, *for* 5 *read* 5^4.

VOL. XL.

Page 8, line 31, *for* 6953 *read* 7126.
,, 46, line 17, *for* $\cot\alpha$ *read* $\cot\theta$.
,, 31, last line, *for* $r_1{}^2 r_2{}^2 \ldots r^2{}_m$ *read* $r_1 r_2 \ldots r_m$.

VOL. XLI.

Page 8, line 12, *for* 7287 *read* 7278.
,, 18, line 42, *for* bx^2 *read* bx.
,, 82, line 7, *for* bx^2 *read* bx.

VOL. XLII.

Page 22, line 1, *for* $H_1 = (h_2 + h_3)$ *read* $H_1 = \frac{1}{2}(h_2 + h_3)$.
,, 102, line 30, *for* $\frac{d}{r}$ *read* $\frac{r}{d}$.
,, 110, line 18, *for* Forming *read* Joining.
,, 115, line 4 from foot, *for* $\frac{16(1-x^2)^2}{4(1-x^2-y^2)}$ *read* $\frac{16(1-x^2)^2}{4y(1-x^2-y^2)}$.

VOL. XLIII.

Page 22, last line, *for* $e^{-a} - a$ *read* $e^{-a} = a$.
,, 36, line 10, *for* Lamina *read* Lemura.
,, 36, line 13, *for* θ *read* A.
,, 40, last line, *for* $f\,b + x$ *read* $f :: b + x$.
,, 44, line 3, *for* $\frac{45\sqrt{15} + 330\sqrt{2}}{23910 + 2700\sqrt{}}$ *read* $\frac{45\sqrt{15} + 330\sqrt{2}}{23910 + 2700\sqrt{30}}$.
,, 44, line 13, *for* $\frac{2x^2 \sin\beta \sin\gamma \cos(\alpha - A)}{}$ *read* $\frac{2x^2 \sin\beta \sin\gamma \cos(\alpha - A)}{a^2}$.
,, 100, line 4, *for* (Fig. 2) *read* (Fig. 5).
,, 121, line 9, *for* μ *read* λ.
,, 123, line 6, *read* Vol. 44, p. 44.
,, 125, line 8, the word "*inverse*" should be at the end of line 7.
,, 137, line 8, omit the commas.
,, 144, line 1, *for* $(-\alpha, -\beta)$ *read* $(+\alpha, +\beta)$.
,, 144, line 28, *for* n *read* x.
,, 146, line 11, *for* $-(p+q)^2$ *read* $1-(p+q)^2$.
,, 147, line 11, *for* the next *read* this.
,, 149, line 14, *for* $+c^2$ *read* $+c^4$.
,, 151, last line, *for* is an *read* and is an.

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